Week 10: The Fundamental Theorem of Calculus and Anti-Derivatives

Remark: Riemann Sums Are Hard

Last week, we introduced the machinery of Riemann sums. They are hard to calculate because we need a special formula for each sum. This week, we introduce a fundamental tool for calculating them which simplifies the process of computing areas.

Theorem: The Fundamental Theorem of Calculus (Version I)

If $f(x)$ is a nice function on $[a, b]$ then:

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

Note: Our textbook calls this "Theorem 5.6: The Net Change Theorem".

"Sketch" of proof.

differentials

actual change

$$\int_a^b \frac{df}{dx} dx = \sum_{k=0}^{N} \frac{df}{dx}(x_k^*) \Delta x$$

$$= \sum_{k=0}^{N} \frac{\Delta f}{\Delta x} \Delta x \quad \leftarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \sum_{k=0}^{N} \Delta f = \sum_{k=0}^{N} \left(f(x_{k+1}) - f(x_k) \right)$$

! Terms cancel in pairs!

$$= \left(\boxed{f(x_1)} - f(x_0)\right) + \left(\boxed{f(x_2)} - \boxed{f(x_1)}\right) + \left(f(x_3) - \boxed{f(x_2)}\right) + \dots + f(x_{N+1}) - f(x_N)$$

$$= f(x_{N+1}) - f(x_0)$$

Definition: Differentials — OpenStax p. 359

The equation $\frac{dy}{dx} = f'(x)$ can be re-written as:

$$dy = f'(x)dx,$$

The terms dy and dx are **differentials**.

Recall that:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

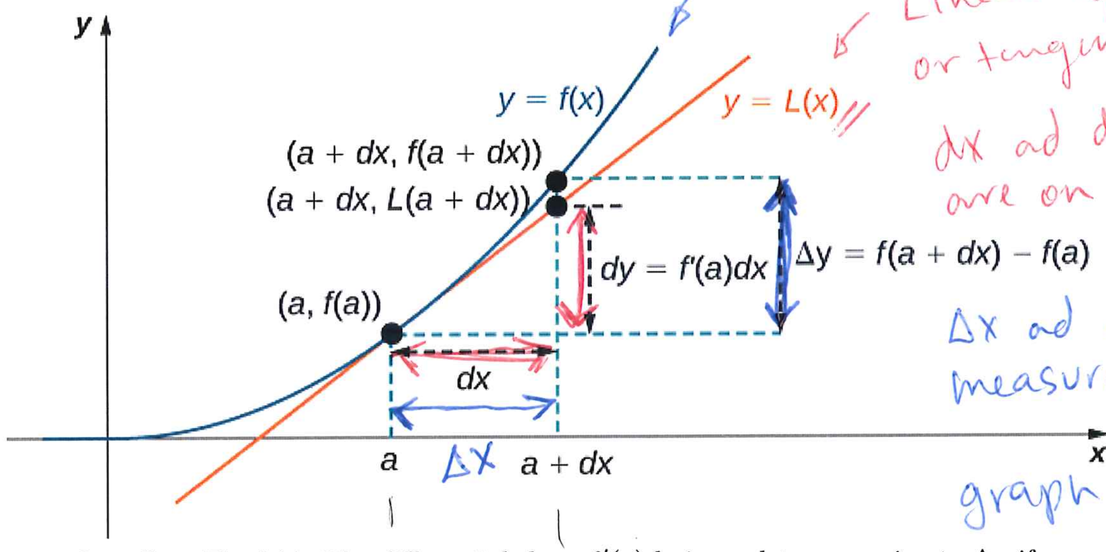
The differentials dy and dx represent "small changes in y and x".

graph of function

Linear approx, or tangent line.

dx ad dy are on tangent line. y = L(x)

Δx ad Δy are measured on graph of y=f(x).



OpenStax Fig 4.11: The differential $dy = f'(a)dx$ is used to approximate Δy if: x increases from $x = a$ to $x = a + dx$.

Example: Find A Differential

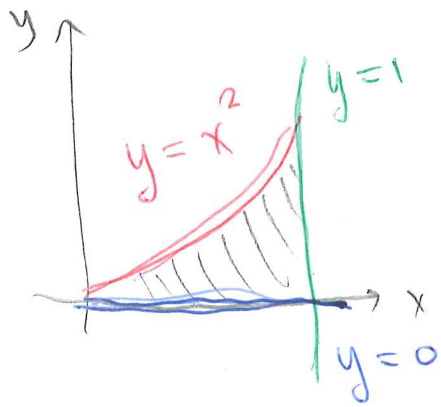
Calculate Δy and dy when $y = x^2 + 2x$, $x = 3$, and $dx = 0.1$

$y = L(x)$ $\left[dy = f'(3)\,dx = (2\cdot 3 + 2)dx = (2\cdot 3+2)\cdot(0.1) = 0.8 \right.$ EASY

$y = f(x)$
$$\left[\begin{aligned} \Delta y &= f(3+0.1) - f(3) \\ &= f(3.1) - f(3) \\ &= \left[(3.1)^2 + 2(3.1)\right] - \left[3^2 + 2\cdot 3\right] = 0.81 \end{aligned} \right.$$
HARD

Example: Calculate An Area (The Hard Problem)

Find the area bounded by $y = x^2$, $y = 0$, and $x = 1$ using the Fundamental Theorem of Calculus.



We want the area:

$$\int_0^1 x^2\,dx = F(1) - F(0)$$

To do so, we need to re-write x^2 so that it looks like a derivative.

$$\int_0^1 x^2\,dx = \int_0^1 \frac{d}{dx}\left[\frac{1}{3}x^3\right]dx = \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}.$$

Fundamental Theorem of Calculus. (v. 1)

? What about constants?

$$\int_0^1 \frac{d}{dx}\left[\frac{1}{3}x^3 + C\right]dx =$$

Yay! The $+C$ constants cancel out.

$$= \left[\frac{1}{3}1^3 + C\right] - \left[\frac{1}{3}0^3 + C\right] = \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}$$

Example: Find An Area **OS §5.4 Q212**

$$\int_0^{\pi/2} x - \sin(x)\, dx$$

What's it asking?

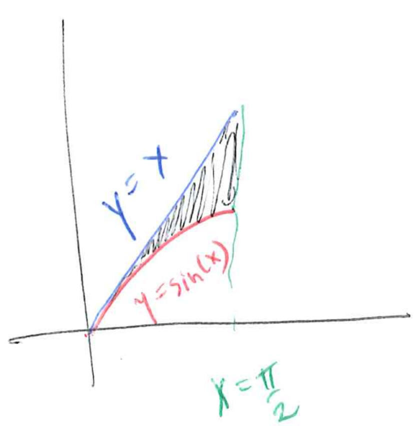
" Find the area bounded by $y=0$, $x=\frac{\pi}{2}$, and $y = x - \sin(x)$. "

Using the fundamental Theorem:

$$\int_0^{\frac{\pi}{2}} x - \sin(x)\, dx = \int_0^{\frac{\pi}{2}} \frac{d}{dx}\left[\frac{1}{2}x^2 + \cos(x)\right] dx$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \cos\left(\frac{\pi}{2}\right)\right] - \left[\frac{1}{2}0^2 + \cos(0)\right]$$

$$= \frac{\pi^2}{8} + 0 - 0 - 1 = \frac{\pi^2}{8} - 1$$



Remark: Integration Requires Us to Undo Differentiation

To apply the Fundamental Theorem of Calculus to a given formula:

$$\int_a^b f(x)\, dx = \int_a^b f(x)\, dx$$

we need to write $f(x) = \dfrac{dF}{dx}$ for some $F(x)$. This means that we need to *undo* the process of differentiation.

Definition: Antiderivatives

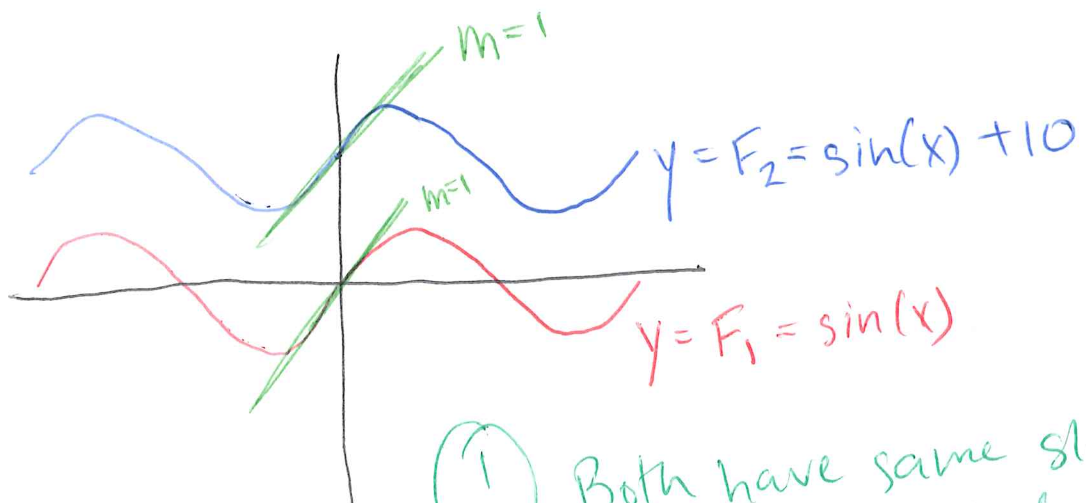
We say that $F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$.
Note: What does $F(x)$ do? It is a function with slope $f(x)$.

Example: Multiple Antiderivatives

Check that $F_1(x) = \sin(x)$ and $F_2(x) = \sin(x) + 10$ are both antiderivatives of $f(x) = \cos(x)$.

$$\frac{d}{dx}[F_1] = \frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[F_2] = \frac{d}{dx}[\sin(x) + 10] = \cos(x) + 0 = \cos(x).$$



! Both have same slope!
And so, antiderivs of cos(x).

Activity

Example: Checking Antiderivatives

Which of the following is an antiderivative of $f(x) = e^{x/2}$? Circle all correct answers.

$\cancel{e^{x/2}}$ $\quad 2e^{x/2} \quad$ $\cancel{2e^{(1+x)/2}}$ $\quad 2e^{x/2}+1 \quad$ $\cancel{e^{x^2/4}}$ (14:21)

$$\frac{d}{dx}\left[e^{x/2}\right] = \frac{1}{2}e^{x/2}$$

$$\frac{d}{dx}\left[2e^{x/2}\right] = 2\cdot\frac{1}{2}e^{x/2} = e^{x/2} \checkmark$$

$$\frac{d}{dx}\left[2e^{(1+x)/2}\right] = 2\cdot\frac{1}{2}e^{(1+x)/2}$$

$$\frac{d}{dx}\left[2e^{x/2}+1\right] = 2\cdot\frac{1}{2}e^{x/2} + 0 = e^{x/2} \checkmark$$

$$\frac{d}{dx}\left[e^{x^2/4}\right] = \left(\frac{2x}{4}\right)e^{x^2/4} = \frac{x}{2}e^{x^2/4}$$

Definition: Integral Notation

We write: $\int f(x)dx = F(x) + C$ to express "$F(x)$ is an antiderivative of $f(x)$". The term $+C$ is called the **constant of integration**. The term $\int f(x)dx$ is called an **indefinite integral**. If have definite bounds, we get a **definite integral**:

$$\int_a^b f(x) = F(b) - F(a)$$

Theorem: Algebra of Indefinite Integrals

We have the following relations:

$$1 \quad \int kf(x)dx = k\int f(x)dx$$

$$2 \quad \int f_1(x) + f_2(x)dx = \int f_1(x)dx + \int f_2(x)dx$$

$$3 \quad \int f_1(x) - f_2(x)dx = \int f_1(x)dx - \int f_2(x)dx$$

$\frac{d}{dx}[\ldots] = \ldots$

$\int \ldots dx = \ldots$

1 If $\frac{d}{dx}[F] = f$ then $\frac{d}{dx}[KF] = K\frac{d}{dx}[F] = Kf$

This gives:

$$K\int f(x)\,dx = \int K f(x)\,dx$$

Example: Powers

If $n \neq -1$ then we have: $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$. (We'll handle $n = -1$ soon.)

We check:

$$\frac{d}{dx}\left[\frac{1}{n+1} x^{n+1} + C\right]$$

$$= \frac{1}{n+1}(n+1)x^{n+1-1} + 0 = \frac{n+1}{n+1} x^n = x^n$$

Yay! The formula works.

Example: A Polynomial Example

Calculate $\int_0^2 x^2 + 3x + 1 \, dx$.

$$\int_0^2 x^2 + 3x + 1 \, dx = \int_0^2 x^2 \, dx + 3\int_0^2 x + \int_0^2 1 \, dx$$

$$= \left[\frac{1}{3}2^3 - \frac{1}{3}0^3\right] + 3\left[\frac{1}{2}2^2 - \frac{1}{2}0^2\right] + \left[2 - 0\right] \quad \text{\# FTC}$$

$$= \frac{1}{3}2^3 + 3 \cdot \frac{1}{2} \cdot 2^2 + 2 = \frac{8}{3} + 6 + 2 = \frac{8}{3} + 8$$

$$= \frac{32}{3}.$$

Theorem: Substitution

The chain rule corresponds to the following antiderivative rule.

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(g(x)) + C$$

Note: This process is often called u-**substitution**.

Example: Applying Substitution

Adjust the integrand as necessary to apply the substitution $u = 4x^2 + 9$ and calculate the indefinite integral.

$$\int \frac{x}{\sqrt{4x^2+9}}\, dx$$

We want to re-write $\frac{x}{\sqrt{4x^2+9}}$ in the format: $f'(g(x))g'(x)$.

$u = g(x) \Rightarrow g'(x) = 8x$

$$\int \frac{x}{\sqrt{4x^2+9}} dx = \frac{1}{8}\int \frac{8x}{\sqrt{4x^2+9}} dx$$

$\leftarrow du = g'(x)dx$

$\leftarrow u = g(x)$

$$= \frac{1}{8}\int \frac{1}{\sqrt{u}} du = \frac{1}{8}\int u^{-1/2} du$$

$$= \frac{1}{8} \frac{1}{1+(-1/2)} u^{1+(-1/2)} + C$$

$$= \frac{1}{8}\frac{1}{\frac{1}{2}} u^{\frac{1}{2}} + C = \frac{1}{4}u^{\frac{1}{2}} + C.$$

$$= \frac{1}{4}(4x^2+9)^{\frac{1}{2}} + C.$$

Replace $u = 4x^2+9$

Goal: $\int \frac{x}{\sqrt{4x^2+9}}\,dx = \frac{1}{4}(4x^2+9)^{1/2} + c.$

We check this with the chain rule:

$$\frac{d}{dx}\left[\frac{1}{4}(4x^2+9)^{1/2} + c\right]$$

$$= \frac{1}{4}\cdot\underbrace{\frac{1}{2}(4x^2+9)^{1/2-1}}_{f'(g(x))}\underbrace{(8x+0)}_{g'(x)}$$

$$= \boxed{\frac{1}{8}}(4x^2+9)^{-1/2}(\boxed{8}x)$$

$$= \boxed{\frac{1}{8}}\frac{\boxed{8}x}{\sqrt{4x^2+9}} = \frac{x}{\sqrt{4x^2+9}}$$

(!) When using substitution, we often need to introduce the boxed terms so that the chain rule works.

Remark: Every Derivative Rule Is Also An Antiderivative Rule

We can convert any derivative rule in to an anti-derivative by reversing the sides of the equality.

$$\frac{d}{dx}[F(x)] = f(x) \iff \int f(x)\, dx = F(x) + C$$

Activity: Class Discussion (5 min)

Produce a table of "anti-derivative rules". (Think about stuff like trigonometry, exponentials, etc.)

(1437)

$f(x)$	$\int f(x)\, dx$
$-\sin(x)$	$\cos(x)$
$\cos(x)$	$\sin(x)$
$\frac{1}{x}$	$\ln(x)$
e^x	e^x
$\sqrt{x}$	$\frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} = \frac{2}{3} x^{\frac{3}{2}}$

$\underline{\text{EX}}$: $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x}}\,dx$

$x^2+2x \rightarrow u = g(x)$

we want to match: $f'(g(x))\,g'(x)$.

we let: $u = g(x) = x^2+2x$

$g'(x) = 2x+2 \leftarrow$ we don't have $2x+2$ but we do have $x+1$

$$\int \frac{x+1}{\sqrt{x^2+2x}}\,dx = \boxed{\frac{1}{2}}\int \frac{\boxed{2}(x+1)}{\sqrt{x^2+2x}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x}}\,dx$$

$u = x^2+2x$

$du = (2x+2)\,dx$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\cdot 2\cdot u^{\frac{1}{2}} + C$$

$$= u^{\frac{1}{2}} + C = (x^2+2x)^{\frac{1}{2}} + C$$

$$= \sqrt{x^2+2x} + C$$

Example: Substitution and Unmatched Terms

$$\int x(1-x)^{99}\,dx$$

(!) Substitution is about chain rule.

We see $(\quad)^{99}$ applied to $1-\square$.

We pick $\boxed{u = 1-x \iff x = 1-u}$ this gives $du = -dx$.

$$\int x(1-x)^{99}\,dx = -\int x\underbrace{(1-x)}_{u}{}^{99}\underbrace{(-dx)}_{du}$$

For any indefinite integral: always undo the substitution.

$$= -\int x(u)^{99}\,du$$

$x = 1-u$

$$= -\int (1-u)u^{99}\,du$$

$$= -\int u^{99} - u^{100}\,du$$

$$= -\left[\frac{1}{100}u^{100} - \frac{1}{101}u^{101}\right] + C$$

$u = 1-x$

$$= -\left[\frac{1}{100}(1-x)^{100} - \frac{1}{101}(1-x)^{101}\right] + C$$

Example: Substitution and Bounds

Evaluate the following definite integral. Change the bounds appropriately when applying substitution.

$$\int_0^1 x\sqrt{2-x^2}\,dx$$

Plug in original bounds to $u(x)$.

Let $u = 2-x^2$. This gives $du = -2x\,dx$.

$$\int_0^1 x\sqrt{2-x^2}\,dx = -\frac{1}{2}\int_0^1 \underbrace{\sqrt{2-x^2}}_{u}\,\underbrace{(-2x\,dx)}_{du}.$$

$$= -\frac{1}{2}\int_{\boxed{1}}^{\boxed{2}} \sqrt{u}\,du = -\frac{1}{2}\int_{\boxed{1}}^{\boxed{2}} u^{1/2}\,du$$

$$= -\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{\boxed{2}}^{\boxed{1}} = -\frac{1}{2}\left[\frac{2}{3}\cdot 1^{\frac{3}{2}} - \frac{2}{3}\cdot 2^{\frac{3}{2}}\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}\boxed{(2-1^2)}^{\frac{3}{2}} - \frac{2}{3}\boxed{(2-0^2)}^{\frac{3}{2}}\right]$$

$x=1$ in $u=2-x^2$

$x=0$ in $u=2-x^2$

This last step is un-necessary. For definite integrals, we don't need to undo the sub. as long as we change bounds appropriately.

Remark: Reversing The Product Rule

So far, we've seen that every derivative rule has a corresponding antiderivative rule. The chain rule becomes substitution. Now, we begin to reverse the sense of the product rule. Notice what this lets us go: We can exchange an integral $f'g$ for fg'. This lets us "move the difficulty of the integral around".

Theorem: Integration by Parts OpenStax Vol II §3.1

The product rule gives us the following:

$$\frac{d}{dx}[fg] = f'g + fg' \iff \int f'g + fg' \, dx = fg + C$$

We rarely have such a nice integrand, and so we re-arrange to get:

$$\int f'g \, dx = fg - \int fg' \, dx + C$$

Example: A Polynomial Times An Exponential

Calculate the following antiderivative: $\int_0^1 xe^x \, dx$.

$$\int_0^1 \underset{g}{x}\, \underset{f'}{e^x} \, dx$$

If $g = x$ then $g' = 1$.

$f' = e^x$ $\quad f = e^x$

$$\int_0^1 e^x \cdot 1 \, dx = \int_0^1 e^x \, dx = [e^x]_0^1 = e^1 - e^0 = e - 1$$

$$= [fg]_0^1 - \int_0^1 fg' \, dx = [xe^x]_0^1 - \int_0^1 e^x \cdot 1 \, dx.$$

$$= [xe^x]_0^1 - [e^x]_{x=0}^{x=1} = [xe^x]_0^1 - [e^1 - e^0]$$

$$= [1e^1 - 0e^0] - [e^1 - e^0] = e^0 = 1.$$

Example: Which Parts?

Which parts should we choose for the following: $\int x^3 \ln(x)\, dx$?
(Don't evaluate the integral, just determine a good choice of parts.)

$$\int x^3 \ln(x)\, dx$$

If we choose $f' = x^3$ and $g = \ln(x)$
we get: $f = \frac{1}{4}x^4$ and $g' = \frac{1}{x}$

This gives: $\int x^3 \ln(x) dx = \frac{1}{4}x^4 \ln(x) - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\, dx$

$$= \frac{1}{4}x^4 \ln(x) - \frac{1}{4}\int x^3 dx$$

Yay! A polynomial.

What would happen if we chose?

$f' = \ln(x)$ $\quad g = x^3$

$f = $? $\quad g' = 3x^2$

We would need to find an anti-deriv of $\ln(x)$.

Example: A Sneaky Choice of Parts

$$\int \ln(x)\ dx$$

$$\int \ln(x)\,dx = \int \underset{f'}{\underset{\uparrow}{1}}\ \underset{g}{\underset{\uparrow}{\ln(x)}}\,dx$$

What is $\int \log_{10}(x)\,dx$?

If $f' = 1$ ad $g = \ln(x)$ we get:

$f = x \qquad g' = \frac{1}{x}$

This gives:

$$\begin{aligned}
\int \ln(x)\,dx &= fg - \int fg'\,dx \\
&= x\ln(x) - \int x\frac{1}{x}\,dx \\
&= x\ln(x) - \int 1\,dx \\
&= x\ln(x) - x + C.
\end{aligned}$$

12:55

Activity: Which Method? (3 min)

Consider the following indefinite integrals. Which method would you try first?
(You don't need to evaluate the indefinite integral. Just pick a tool and say how you'd apply it.)

1. $\int x^2 + 2x \, dx$ ← direct.
2. $\int xe^{-x^2} \, dx$ ← Substitution $u = -x^2$ and $du = -2x\,dx$
3. $\int x^2 + \cos(2x) \, dx$ ← direct + substitution $u = 2x$ ad $du = 2dx$.
4. $\int x^2 \cos(x) \, dx$ ← parts: $f' = \cos(x)$ ad $g = x^2$
5. $\int e^x \cos(x) \, dx$ ← parts: $f' = e^x$ and $g = \cos(x)$

Currently our methods are: direct, substitution, or parts.

Remark: Linear Approximations

Now we're going to go on a quick detour through the land of linear approximations. This is *really* just taking a tangent line. Nothing new here. This detour is a refresher to provide context for differentials.

Definition: Linear Approximations OpenStax p. 355

The **linear approximation** to $f(x)$ at $x = a$ is:

$$L(x) - f(a) = f'(a)(x - a)$$

Notice:

- This is the point-slope format of a line.
- The line has slope $m_{tan} = f'(a)$.
- The line passes through $(a, f(a))$.

Activity: Try It Out (2 min)

(13:15)

Find the tangent line of $f(x) = \sqrt{x}$ at $a = 4$.

$$m_{tan} = f'(a) = \frac{1}{2\sqrt{4}} = \frac{1}{2\cdot 2} = \frac{1}{4}$$

$$L(x) - f(a) = \frac{1}{4}(x-a)$$

$$L(x) - \sqrt{4} = \frac{1}{4}(x-4)$$

$$L(x) - 2 = \frac{1}{4}(x-4)$$

$$\Rightarrow y = L(x) = \frac{1}{4}(x-4) + 2 = \frac{1}{4}x - 1 + 2 \boxed{= \frac{1}{4}x + 1}$$

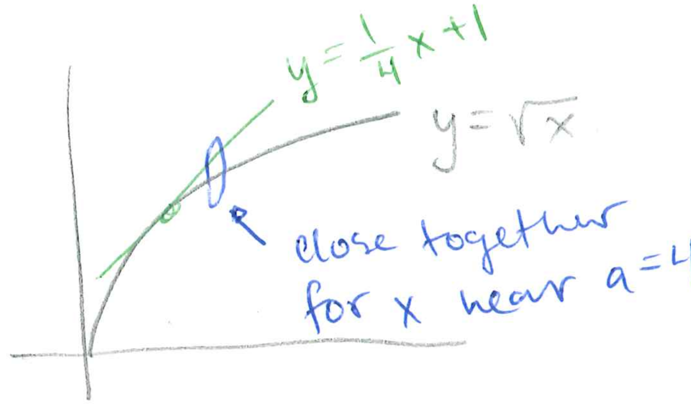
Example: Approximate Using A Tangent Line

Use the tangent line $y = \frac{1}{4}x + 1$ of $y = \sqrt{x}$ at $a = 4$ to approximate $\sqrt{4.01}$.

$$y = \frac{1}{4}(4.01) + 1 \approx \sqrt{4.01} \approx 2.0025$$

$$= 1.0025 + 1$$

$$= 2.0025$$



$$\frac{1}{4}(4.01) + 1 = \underline{2.0025} \approx \sqrt{4.01} = \underline{2.00249}\ldots$$

We match the first three decimal places!