Week 8: Curve Sketching, Differentials, and Approximation

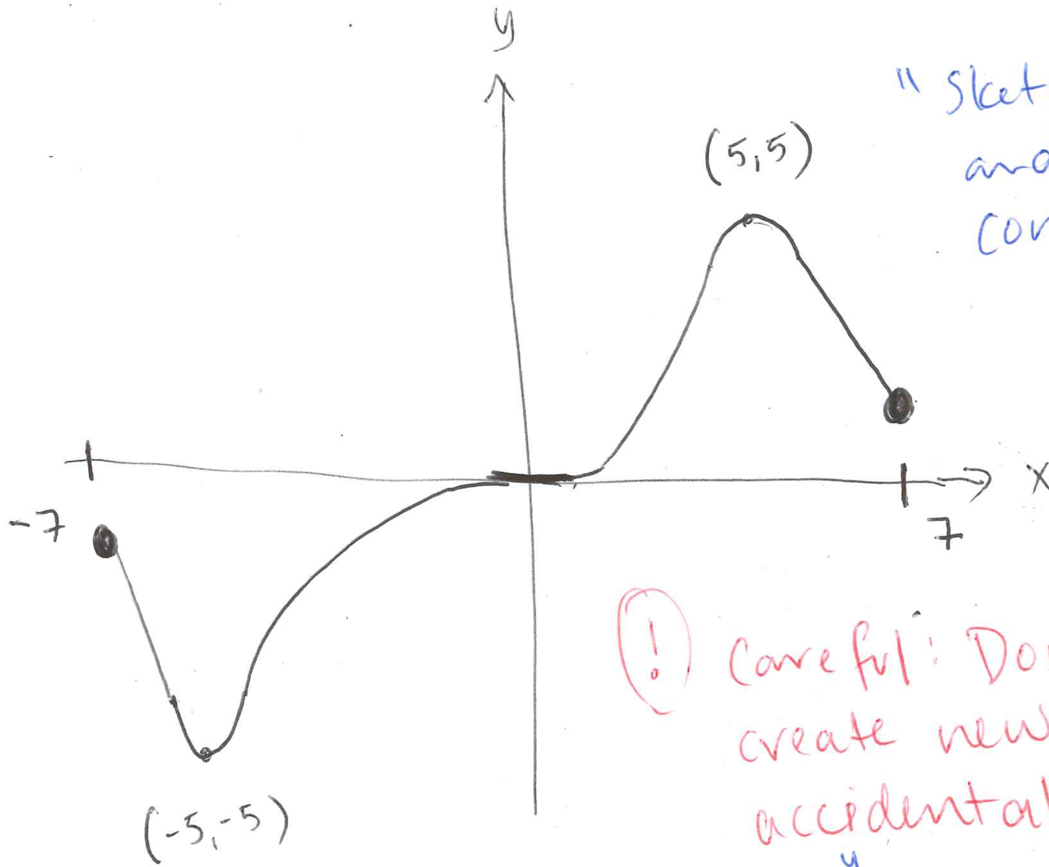
Remark: Mish-Mash Warning

This week has a bit of everything. We're going to do a bunch of small things.

Question: Sketching from Given Data

Draw a graph $f(x)$ defined on the interval $[-7, 7]$ with all of the following properties:

1. A local maximum at $f(5) = 5$
2. A local minimum at $f(-5) = -5$
3. A critical point that is not a maximum or minimum at $x = 0$.



"Sketch features and then connect the dots."

(!) Careful: Don't create new features accidentally.

oops! New local max & min.

Definition: Rational Functions — OpenStax p. 47

A **rational function** is a function $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.
Rational functions are helpful because they:

- Often have vertical asymptotes.
- Easy to analyze end-behaviour.
- Complicated curve sketches.

Example: A Rational Curve Sketch

Sketch the graph of $f(x) = \frac{1}{1+x^2}$.

- Find the x and y intercepts.
- Find the critical points of $f(x)$.
- Make a sign chart for $f'(x)$.
- Classify the critical points.
- Locate points of inflection.
- Analyze asympotes and end-behaviour.

Intercepts.

$x = 0 \Rightarrow y = \dfrac{1}{1+0^2} = 1$

y-intercept is $(0,1)$

If $\dfrac{1}{1+x^2} = 0$ then $1 = 0$.

Oh no!

There is no x-intercept.

Critical points.

$$\frac{dy}{dx} = \frac{\frac{d}{dx}[1](1+x^2) - 1 \cdot \frac{d}{dx}[1+x^2]}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2} = 0$$

$\dfrac{dy}{dx} = 0 \iff x = 0$

We get one critical point at $x = 0$.

$$\left.\frac{dy}{dx}\right|_{x=-1} = \frac{-2(-1)}{(1+(-1)^2)^2} = \frac{2}{4} = \frac{1}{2} > 0$$

Sign Chart

Interval	Test Point	Sign	Conclusion
$(-\infty, 0)$	$x = -1$	$+$	$x = 0$ local max
$(0, \infty)$	$x = 1$	$-$	

$$\frac{dy}{dx} = \frac{-2x}{(1+x^2)^2} = 0$$

$$\Longleftrightarrow \quad -2x = 0 \cdot (1+x^2)^2 = 0$$

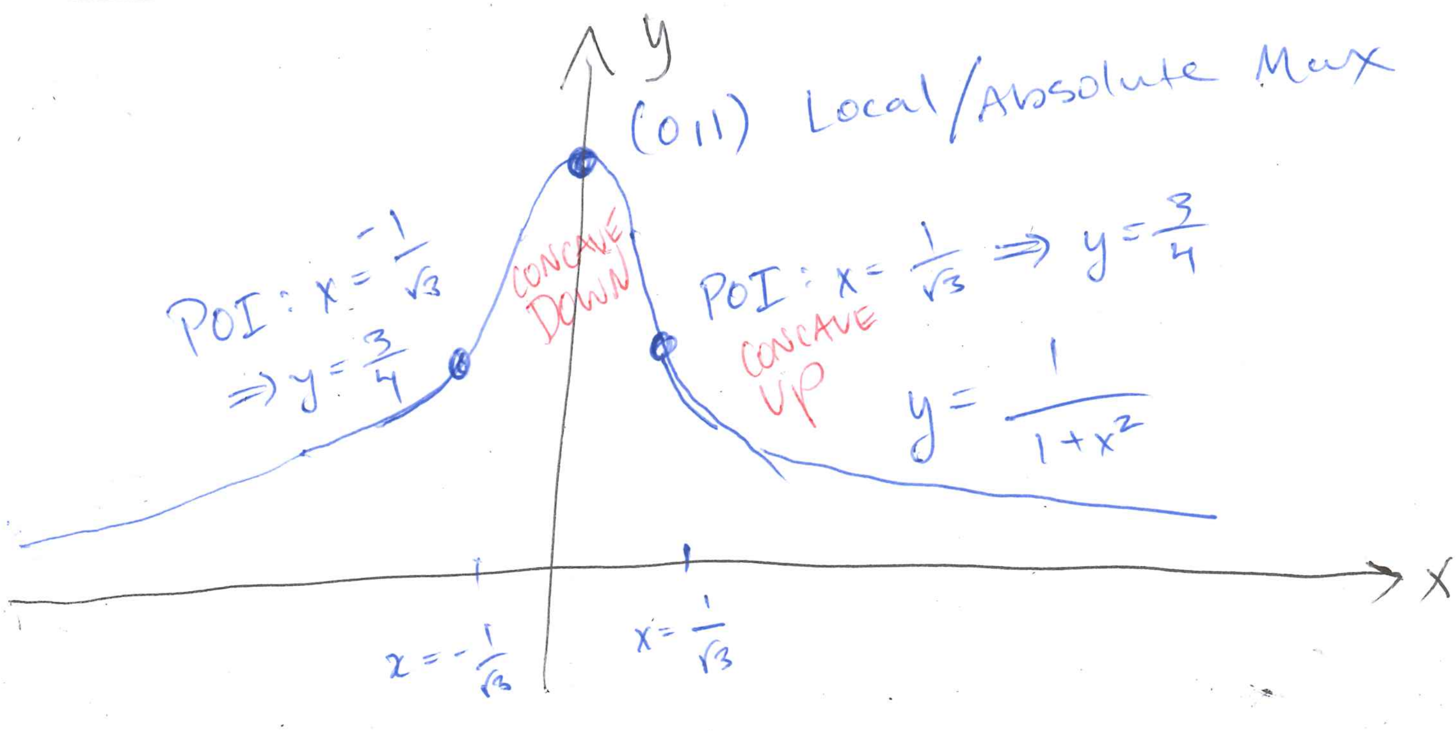
$$\Longleftrightarrow \quad -2x = 0$$

$$\Longleftrightarrow \quad (-2x)\left(-\frac{1}{2}\right) = 0\left(-\frac{1}{2}\right)$$

$$\Longleftrightarrow \quad x = 0$$

? What's a critical point

$\frac{dy}{dx} = 0$ OR $\frac{dy}{dx} =$ undefined.



(This page continues the example: Sketch the graph of $f(x) = \frac{1}{1+x^2}$.)

Points of inflection

$$\frac{dy}{dx} = \frac{-2x}{(1+x^2)^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dx}[-2x](1+x^2)^2 - (-2x)\frac{d}{dx}[(1+x^2)^2]}{(1+x^2)^4}$$

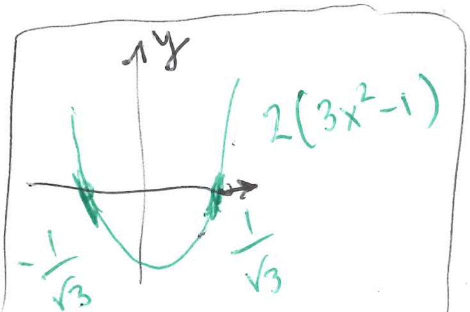
$$= \frac{-2\underline{(1+x^2)}^2 + 2x\left(2\underline{(1+x^2)}(2x)\right)}{\underline{(1+x^2)}^4}$$

$$= \frac{(1+x^2)\left[-2(1+x^2) + (2x)(2x)2\right]}{(1+x^2)^4} = \frac{-2-2x^2+8x^2}{(1+x^2)^3}$$

$$= \frac{-2+6x^2}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3}$$

$2(3x^2-1) \leftarrow$ Changes sign twice.

$(1+x^2)^3 \leftarrow$ Denom always positive.



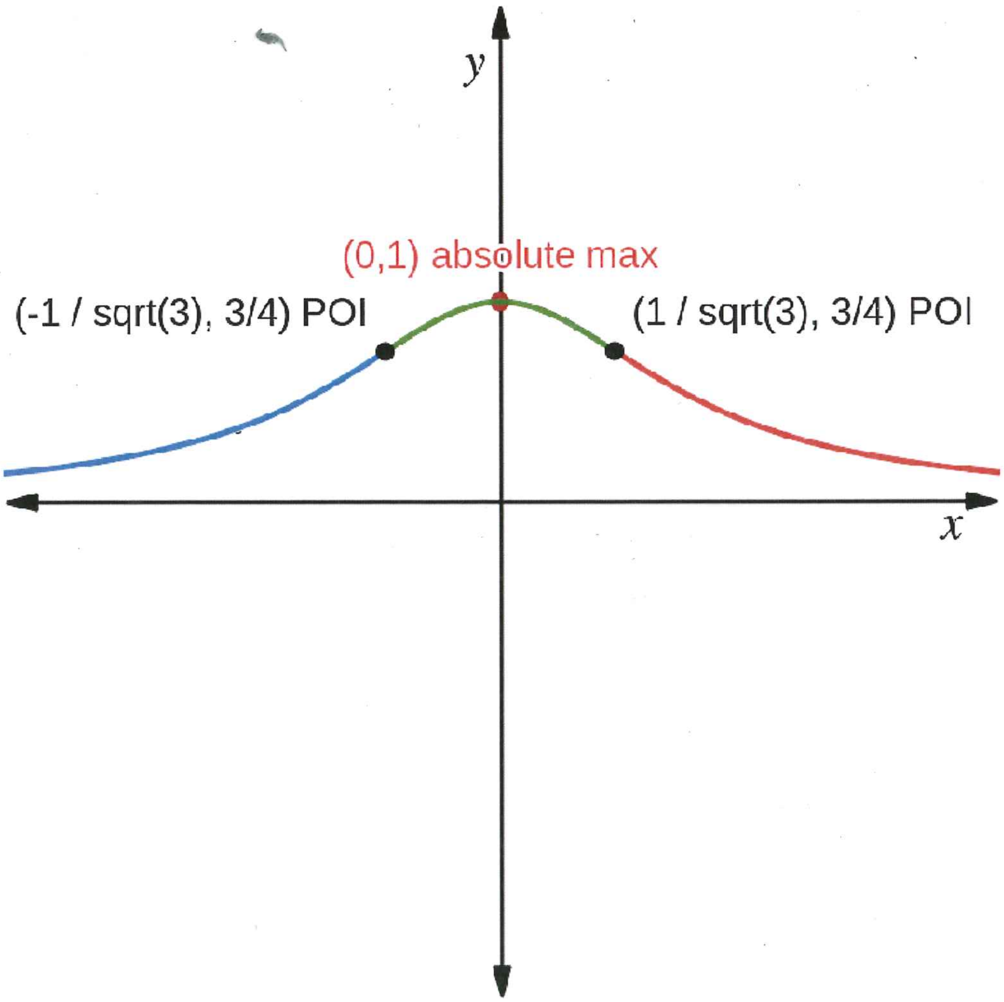
Just the numerator

$$3x^2 - 1 = 0$$
$$\Leftrightarrow 3x^2 = 1$$
$$\Leftrightarrow x^2 = \frac{1}{3}$$
$$\Leftrightarrow x = \pm\frac{1}{\sqrt{3}}$$

We get: points of inflection at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$.

List of Features:

- ✓ y-intercept $(0,1)$ and no x-intercept.
- ✓ local max at $x=0$
- points of inflection at $x = \pm\frac{1}{\sqrt{3}}$
- domain is $\mathbb{R}$.



$y = \frac{1}{1+x^2}$

NO REAL ROOTS

https://www.desmos.com/calculator/brz4ngiuiv

Definition: Vertical Asymptotes — OpenStax p. 149

Consider a function $f(x)$. We say $x = a$ is a **vertical asymptote** if any of the following occur:

1. $\lim\limits_{x \to a^+} f(x) = \pm\infty$
2. $\lim\limits_{x \to a^-} f(x) = \pm\infty$
3. $\lim\limits_{x \to a} f(x) = \pm\infty$

(!) If $x=a$ is an asymptote then x is $\underbrace{\text{NOT}}_{\text{usually}}$ in the domain.

Definition: Horizontal Asymptotes — OpenStax p. 408 (!)

A function $f(x)$ has a **horizontal asymptote** $y = L$ if either of the following occur:

1. $\lim\limits_{x \to \infty} f(x) = L$
2. $\lim\limits_{x \to -\infty} f(x) = L$

Example: Find Vertical Asymptotes

Determine where $y = \dfrac{x-1}{x^2 - 3x + 2}$ has vertical asymptotes.

We factor numerator and denominator:

$$y = \frac{x-1}{(x-1)(x-2)} = \frac{1}{x-2} \quad (\text{for } x \neq 1)$$

Roots of the factored denominator after cancellation

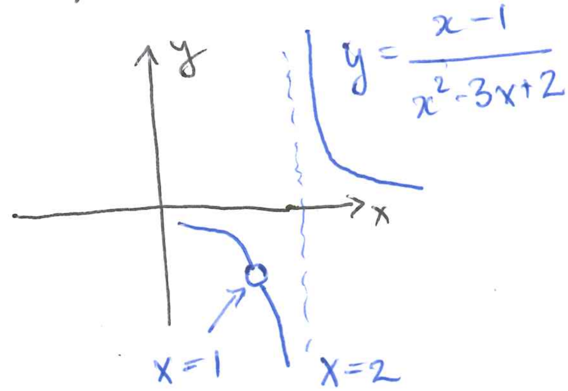
This gives $\lim\limits_{x \to 2^+} = \infty$ and $\lim\limits_{x \to 2^-} = -\infty$.

There is a vertical asymptote at $x = 2$.
There is a hole at $x = 1$.

At $x=1$ we get "$\frac{0}{0}$" (undefined)

However,

$$\lim_{x \to 1} \frac{x-1}{x^2-3x+2} = -1 \neq \pm\infty$$



Therefore $x=-1$ is NOT an asymptote.

Example: Find Horizontal Asymptotes

Determine the horizontal asymptotes of $y = \dfrac{x-1}{x^2-3x+2}$.

We calculate

$$\lim_{x\to\infty} \frac{x-1}{x^2-3x+2} = \text{"}\frac{\infty}{\infty}\text{"}$$

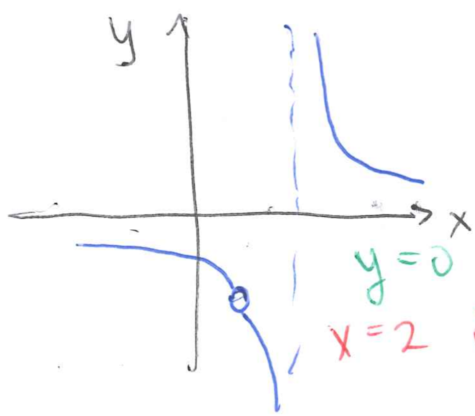
Factor highest power from top-and-bottom.

$$= \lim_{x\to\infty} \frac{x^2\left(\frac{1}{x}-\frac{1}{x^2}\right)}{x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x^2}}{1-\frac{3}{x}+\frac{2}{x^2}}$$

$$= \frac{0-0}{1-0-0} = \frac{0}{1} = 0$$

Similarly, $\lim_{x\to-\infty} \dfrac{x-1}{x^2-3x+2} = 0$.

Therefore, $y=0$ is a horizontal asymptote.



$y=0$ is a horizontal asymp.

$x=2$ is a vertical asymp.

$$y = \frac{10x^2 + 1}{12x^2 + x + 1}$$

$$\lim_{x \to \infty} \frac{10x^2 + 1}{12x^2 + x + 1} = \lim_{x \to \infty} \frac{x^2\left(10 + \frac{1}{x^2}\right)}{x^2\left(12 + \frac{1}{x} + \frac{1}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{10 + \frac{1}{x^2}}{12 + \frac{1}{x} + \frac{1}{x^2}} = \frac{10 + 0}{12 + 0 + 0} = \frac{10}{12} = \frac{5}{6}$$

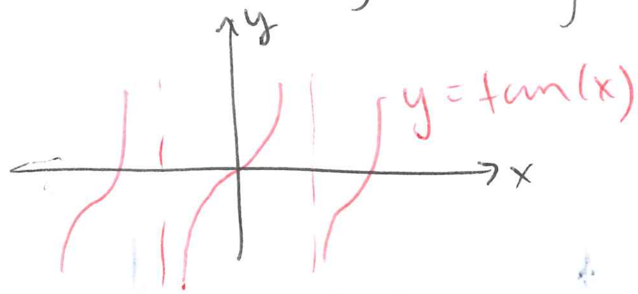
Activity: Asymptotic Trickiness (5 min)

Find an example or justify each of the following claims about asymptotes:

- True! A function can have many vertical asymptotes. ∞ many! counter-example
- A function can only have two horizontal asymptotes. we can have at most two.
- A function can cross its horizontal asymptotes.
- A function cannot cross its vertical asymptotes.

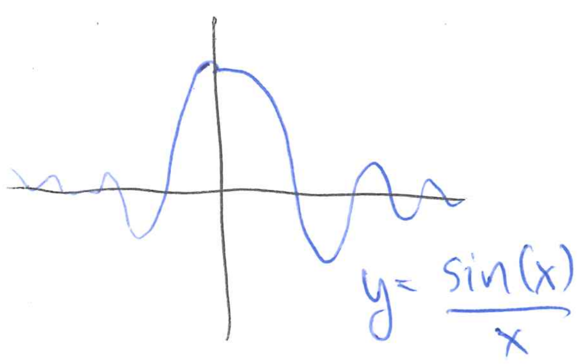
Sketch your examples using Desmos. 14:30

(1) $y = \tan(x)$ has infinitely many asymptotes



(2) A function can have zero, one, or two. horizontal asymptotes.

(3) A function CAN cross its horz. asymptotes



Example: A Curve Sketch with Asymptotes

Sketch the curve $y = \dfrac{1}{1-x^2}$ assuming:

$$\frac{dy}{dx} = \frac{2x}{(1-x^2)^2} \qquad \frac{d^2y}{dx^2} = \frac{6x^2+2}{(1-x^2)^3}$$

Intercepts

$x=0 \Rightarrow y=1$ y-intercept is: $(0,1)$.

No x-intercepts.

Critical points.

$\boxed{\frac{dy}{dx} = 0 \Leftrightarrow x = 0}$ OR $\frac{dy}{dx}$ undefined. $\Leftrightarrow 1-x^2=0 \Leftrightarrow x = \pm 1$

Sign Chart

Interval	Test Point	Sign	Conclusion
$(-\infty, -1)$	$x=-10$	$-$	
			$\longrightarrow$ $x=-1$ not max/min
$(-1, 0)$	$x=-\frac{1}{2}$	$-$	
			$\longrightarrow$ $x=0$ min
$(0, 1)$	$x=\frac{1}{2}$	$+$	
			$\longrightarrow$ $x=1$ neither max/min
$(1, \infty)$	$x=10$	$+$	

Points of Inflection

$$\frac{d^2y}{dx^2} = \frac{6x^2+2}{(1-x^2)^3}$$

$6x^2+2 \leftarrow$ Always positive

$(1-x^2)^3 \leftarrow$ Changes signs when $1-x^2=0$

$\Leftrightarrow 1=x^2$

$\Leftrightarrow x=\pm 1$

Asymptotes

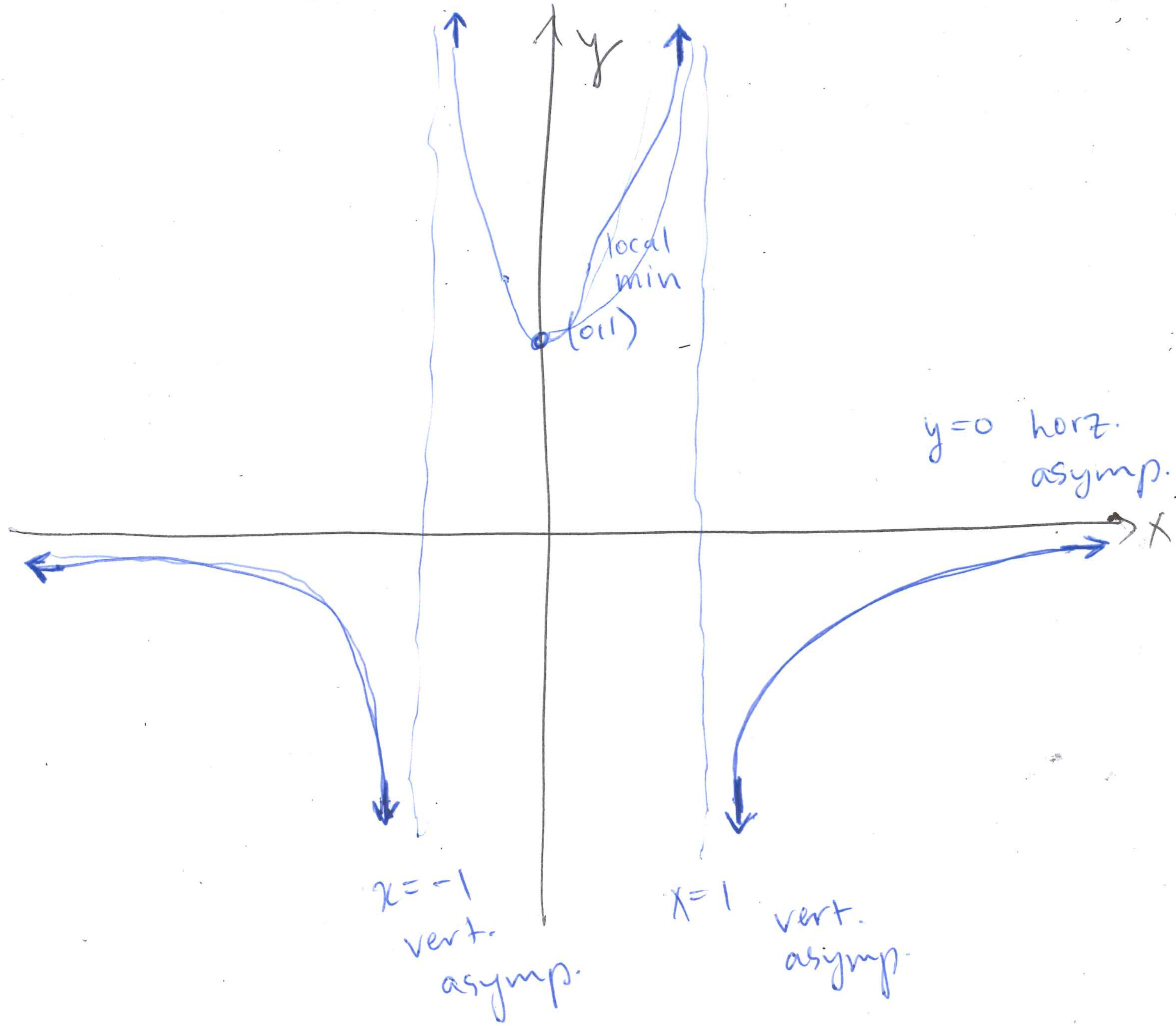
$$y = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)}$$

$$\boxed{\begin{aligned} &\lim_{x \to 1^+} = -\infty \\ &\lim_{x \to 1^-} = \infty \end{aligned}} \Rightarrow x = 1 \text{ is a vert. asymp.}$$

$$\lim_{x \to -1^-} = -\infty \Rightarrow x = -1 \text{ is a vert asymp.}$$

$$\lim_{x \to -1^+} = \infty$$

$$\boxed{\lim_{x \to \infty} \frac{1}{1-x^2} = 0^- \Rightarrow y = 0 \text{ is a horz. asymp.}}$$



y
local
min
(0,1)
y=0 horz.
asymp.
x
x=-1
vert.
asymp.
x=1
vert.
asymp.

Remark: Log-Log and Semi-Log Plots

We introduce the notions of log-log graphs and semi-log graphs. These are (probably) the first new topics in the course for some students. Unfortunately, they're in the course description AND the textbook doesn't have any material on them.

Definition: Log-Log and Semi-Log Plots

To graph a **semi-log** graph:

- Calculate a list of points (x, y) on the graph.
- Plot each point: draw a point (x, y) which is x units from the y-axis and $\log(y)$ units from the x-axis.

To graph a **log-log** graph:

- Calculate a list of points (x, y) on the graph.
- Plot each point: draw a point (x, y) which is $\log(x)$ units from the y-axis and $\log(y)$ units from the x-axis.

Activity: Class Discussion (3 min)

The instructions above describe how to plot log-log and semi-log graphs. Formulate a similar set of instructions for plotting a standard cartesian graph like in highschool.

12:48

- Make a table of values. (x, y).
- Plot each point: draw a point (x, y) which is $\boxed{x}$ units from y-axis ad $\boxed{y}$ units from x-axis.
- Connect the dots.

Example: Make a Semi-Log Plot

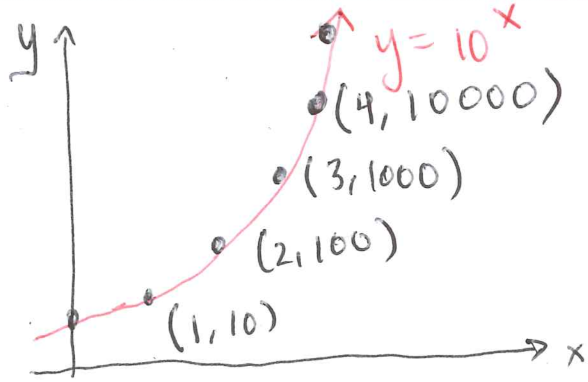
Consider the function $y = 10^x$.

1. Calculate five points where $x = 1, 2, 3, 4, 5$ on this graph.
2. Plot the five points from part (1) on a standard cartesian graph. ("normal")
3. Plot the five points from part (1) on a semi-log graph.

(1)

x	$y = 10^x$
1	10
2	100
3	1000
4	10 000
5	100 000

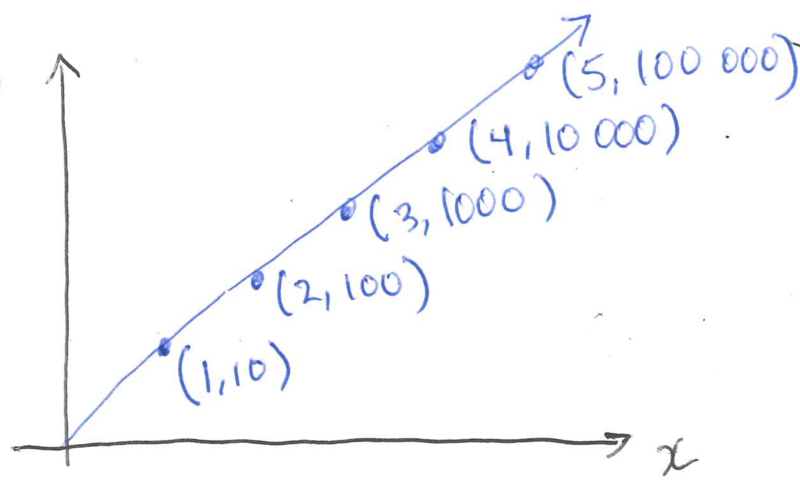
(2)



Notice

This is a curve.

(3)



x	$\log(y)$
1	$\log(10) = 1$
2	$\log(100) = 2$
3	$\log(1000) = 3$
4	4
5	5

The point (x, y) is $\log(y)$ units above x-axis.

Notice: This is a straight line!

~~The graph of~~ The semi-log graph of $y = 10^x$ looks like "$y = x$".

Example: Make a Log-Log Plot

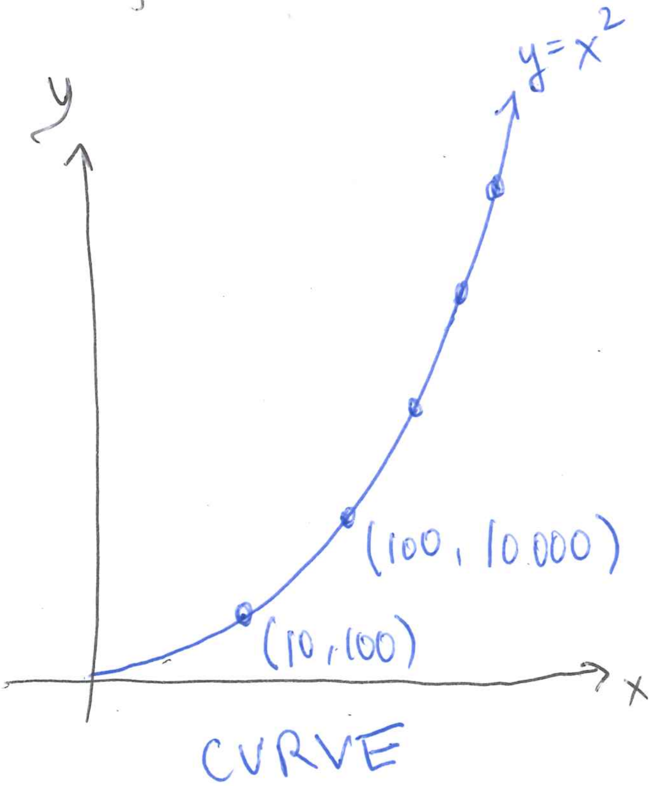
Consider the graph $y = x^2$. $x = 10, 100, 1000, 10000, 100\,000$

1. Calculate five points where ~~$x = 1, 2, 3, 4, 5$~~ on this graph.
2. Plot the five points from part (1) on a standard cartesian graph.
3. Plot the five points from part (1) on a log-log graph.

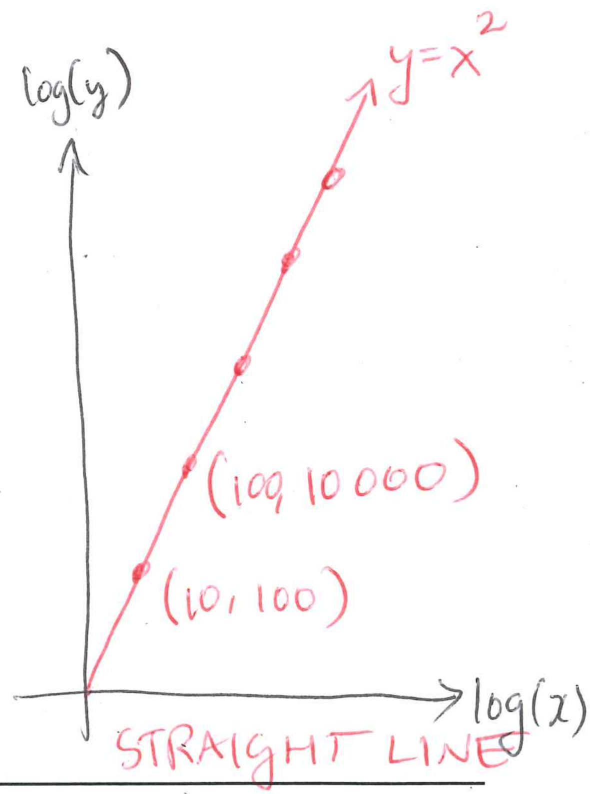
(1) ~~x log~~

x	y	log x	log y
10	$10^2 = 100$	$1 = \log(10)$	$\log(100) = 2$
100	$100^2 = 10\,000$	$2 = \log(100)$	$\log(10000) = 4$
1000	$1000^2 = 1\,000\,000$	3	6
10 000	$10\,000^2 = 1\,00\,00\,00\,00$	4	8
100 000	$100\,000^2 = 1\,00\,00\,00\,00\,00$	5	10

(2)



(3)



Theorem: The Power of Log Graphs

If $f(x)$ is an exponential function of the form $f(x) = Ae^{Bx}$ then the semi-log graph of $y = f(x)$ appears to be a straight line. If $g(x)$ is a power function of the form $g(x) = Ax^B$ then the log-log graph of $y = g(x)$ appears to be a straight line.

Big Idea: Log graphs help us analyze data.

Example: Kepler's Third Law

Make a log-log plot of the following points (P, A) for each planet.

Planet	Orbital Period (P)	Axis (A)
Mercury	0.241	0.39
Venus	0.615	0.72
Earth	1.00	1.00
Mars	1.88	1.52
Jupiter	11.8	5.2
Saturn	29.5	9.54
Uranus	84.0	19.18
Neptune	165	30.06
Pluto	248	39.44

$\longrightarrow \log(P) = \log(A) = 0$

Staring at the log-log graph in Desmos:
we get: $A = P^{2/3}$

The power-law relationship comes from the fact that the relation is linear in log-log coordinates.

https://www.desmos.com/calculator/b54bankl0q

Example: Deduce a Power Law from Given Data

Use the log-log plot to find a relation of the form $A \approx P^c$ between A and P.
History: Kepler worked on this problem for decades. It is a major accomplishment.

Let us take two points from the above log-log graph, *i.e.*, Earth and Neptune.

For Earth, $(\log_{10}(P), \log_{10}(A)) = (0, 0)$.
For Neptune, $(\log_{10}(P), \log_{10}(A)) = (1.924, 1.282)$.

We know that slope of log-log graph is the exponent of power law relationship. Therefore, the slope of log-log graph is,

$$m = \frac{1.282 - 0}{1.924 - 0} = \frac{1.282}{1.924} = 0.666 \approx \frac{2}{3}.$$

Hence, the relationship between P and A is:

$$A \approx P^{2/3}.$$

We have $\log(A) = \frac{2}{3}\log(P) \iff A = P^{2/3}$

$\boxed{\text{Logarithms: } 10^{\log(x)} = x}$

We take powers of both sides

$$10^{\log(A)} = 10^{\frac{2}{3}\log(P)}$$

$$\iff 10^{\log(A)} = \left(10^{\log(P)}\right)^{\frac{2}{3}}$$

$$\iff A = P^{\frac{2}{3}}$$