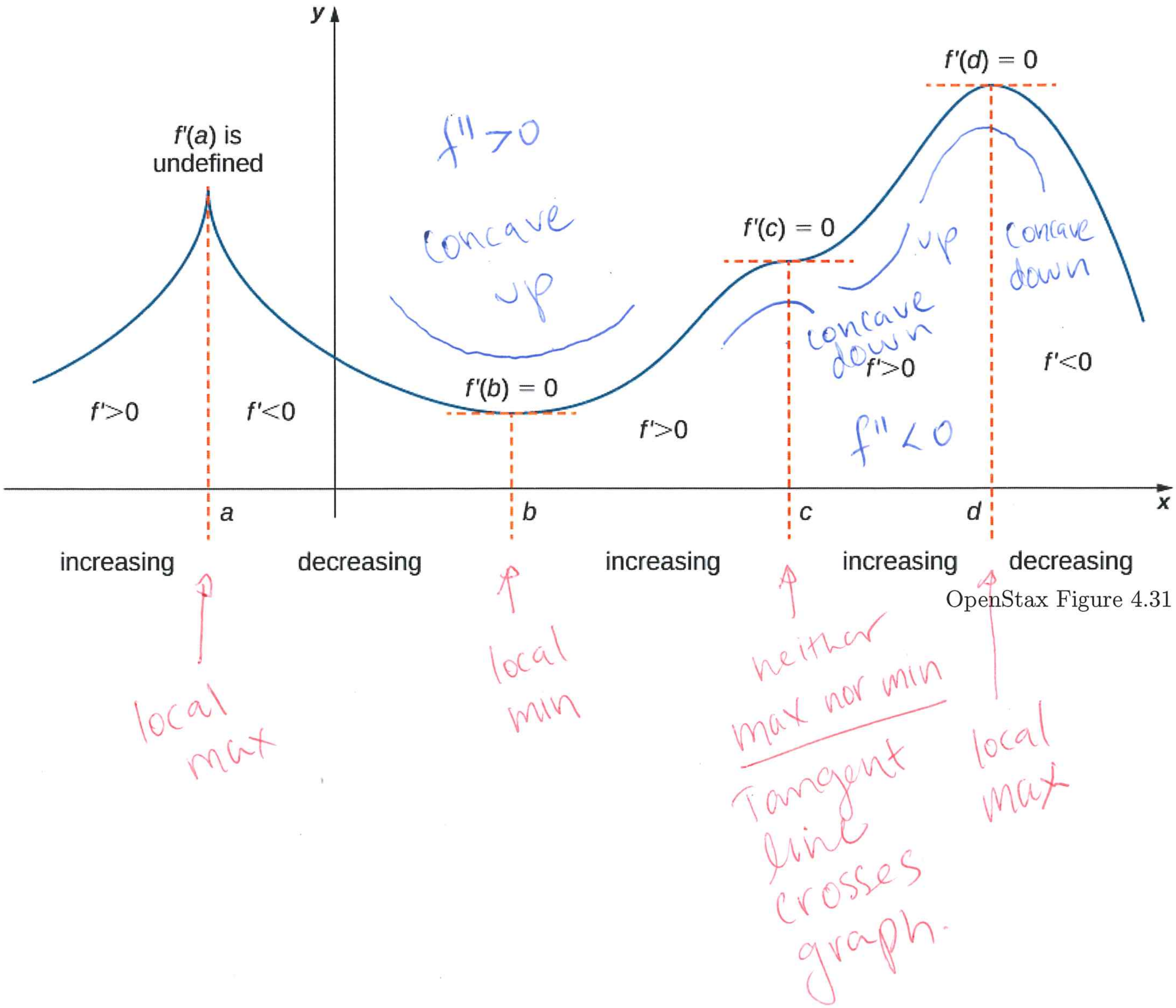


OpenStax Figure 4.31

- Point of inflection:
  Tangent line crosses graph
  $\Longleftarrow$ Concave up to concave down (or vice versa)

Question: Building a Sign Chart

Suppose that $f'(x) = 2x^2(2x - 9)$. Find the locations of all local extrema.

This is the deriv. of f(x)

Interval	Test Point	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	$x = -1$	$-$	
$(0, 9/2)$	$x = 1$	$-$	$x=0$ neither min nor max
$(9/2, \infty)$	$x = 10$	$+$	$x=\frac{9}{2}$ is a min.

Find the critical points.

$f'(x) = 2x^2(2x-9) = 0 \Rightarrow x=0$ OR $x=\frac{9}{2}$.

Create intervals between critical points.

$(-\infty, 0) \cup (0, \frac{9}{2}) \cup (\frac{9}{2}, \infty)$

Pick ~~the test~~ test points.

$-1 \in (-\infty, 0) \Rightarrow f'(-1) = -22 < 0$

$1 \in (0, \frac{9}{2}) \Rightarrow f'(1) = -14 < 0$

$10 \in (\frac{9}{2}, \infty) \Rightarrow f(10) = 2200 > 0$

Nice easy to calculate numbers.

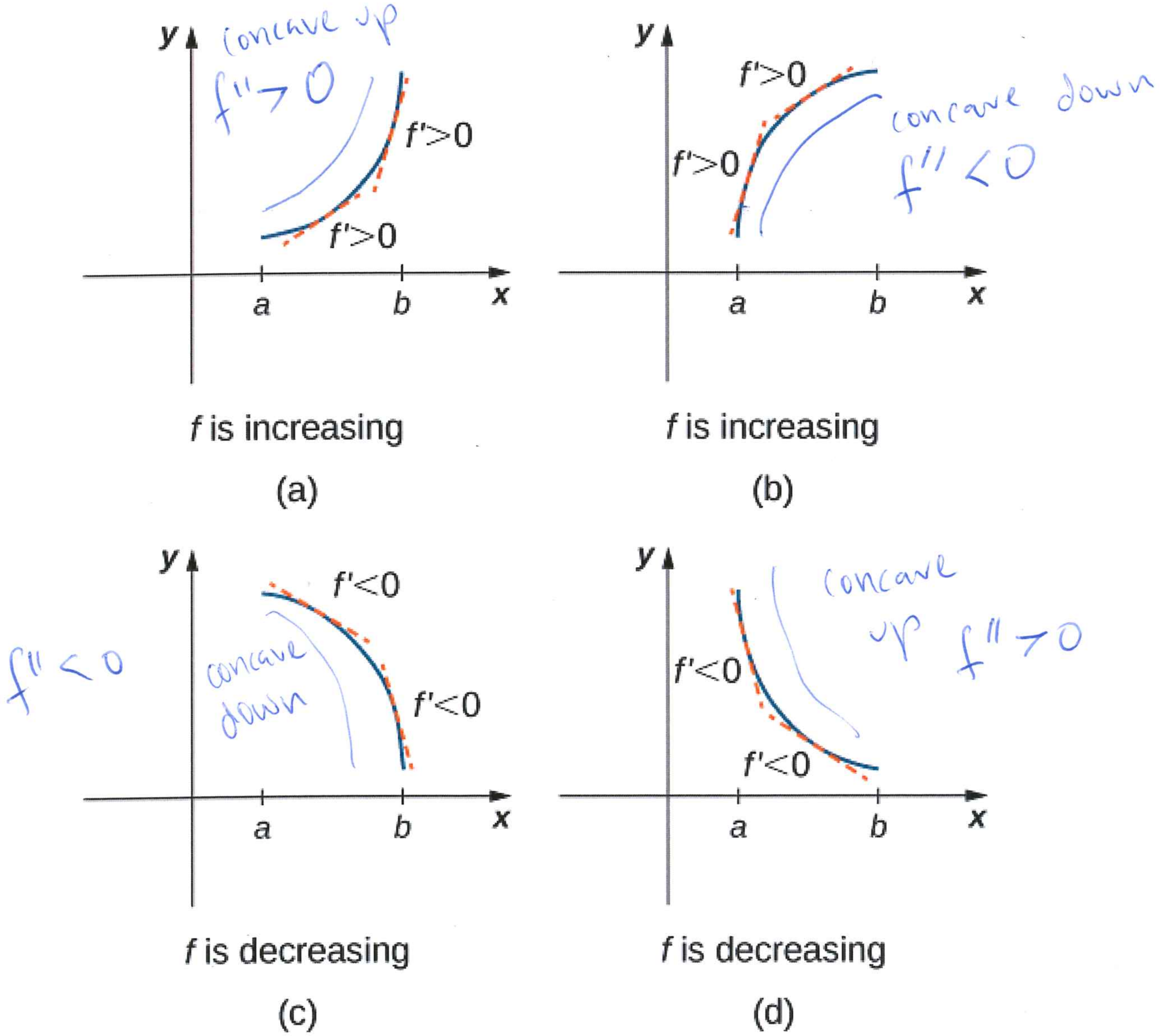
Therefore $x=\frac{9}{2}$ is a local minimum.

Problem-Solving Process:

- Create intervals using the critical points $x = c$.
- Pick a "test point" in each interval.
- Evaluate the derivative at the test point.
- Conclude whether $f(x)$ is increasing / decreasing on each interval.
- Form a chart of this information.

Activity: Increasing and Decreasing: What do you notice?

What do you notice about the graphs below?



OpenStax Figure 4.30

We need some new vocabulary!

Definition: Higher Derivatives

If $f(x)$ is a function then $f'(x)$ is its **first derivative.** ← f' is also a function!
The **second derivative** of $f(x)$ is:

$$f''(x) = \frac{d}{dx}\left[f'(x)\right] = \text{“the derivative of the first derivative”}$$

We can also write:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

In general, the n'th derivative is:

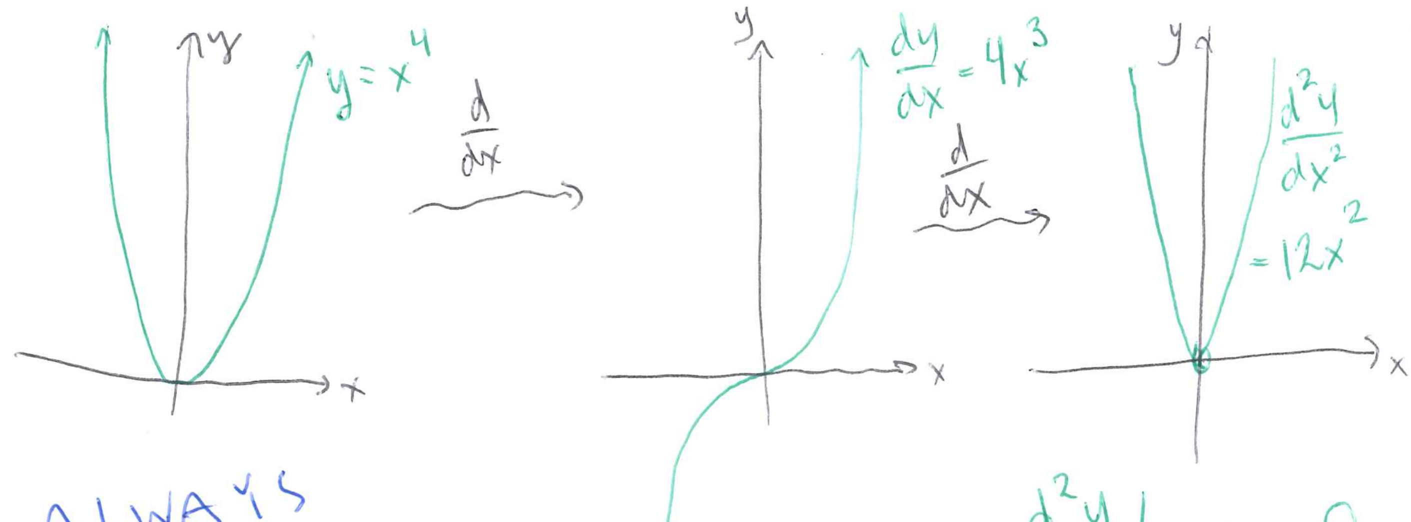
$$f^{(n)}(x) \qquad \frac{d^ny}{dx^n}$$

Definition: Concavity OpenStax p. 395

If f' is increasing, then we say that f is **concave up**.
If f' is decreasing, then we say that f is **concave down.**

Theorem: Second Derivatives Detect Concavity

If $f'' > 0$ then f is concave up. If $f'' < 0$ then f is concave down.
Warning: $f''(a) = 0$ does not imply a change of concavity.
$y = x^4$ has $\frac{d^2y}{dx^2} = 4 \cdot 3 \cdot x^2 = 0$ at $x = 0$ but $y = x^4$ is always concave up.

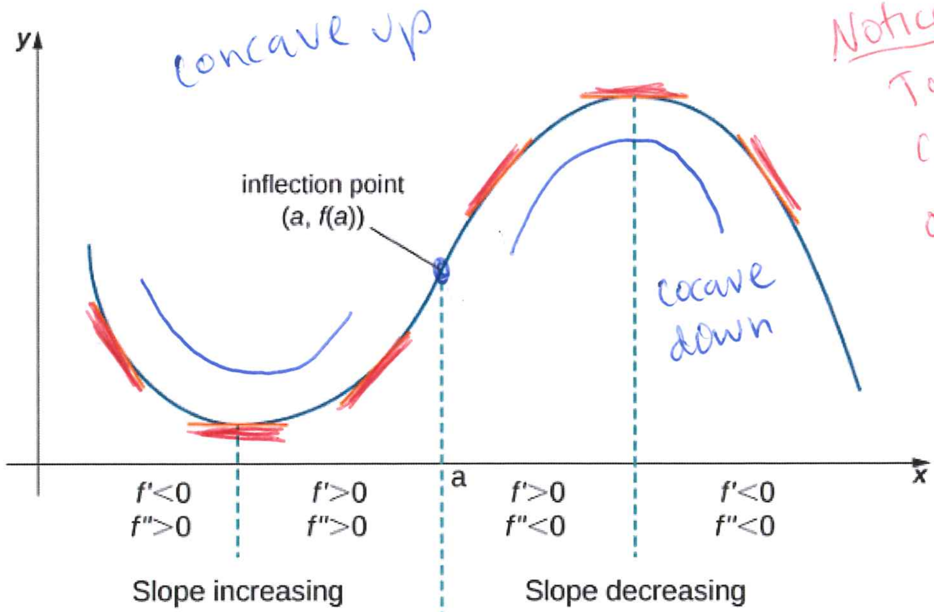


ALWAYS CONCAVE UP

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 0$$

Definition: Points of Inflection

If f changes concavity at $x = a$ then $(a, f(a))$ is a **point of inflection**. (POI)

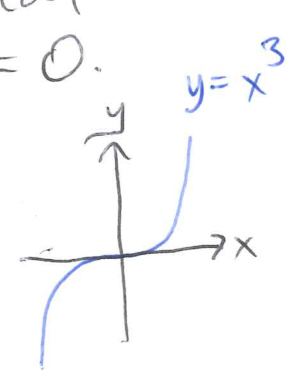


Notice: Tangent line changes side of graph at POI

OpenStax Fig 4.35

Notice: In this case, $x=a$ is a POI but it is NOT a critical point. The graph suggests $f'(a) > 0$.

The function $y = f(x) = x^3$ has both a critical point AND a point of inflection at $x = 0$.



CRITICAL POINT:

$f'(x) = 3x^2 = 0$

$\Rightarrow x = 0$

POINT OF INFLECTION

$f''(x) = 6x$

$x < 0 \Rightarrow f''(x) < 0$

$x > 0 \Rightarrow f''(x) > 0$

Theorem: The Second Derivative Test — OpenStax p. 400

Suppose $f'(c) = 0$ and $f''(x)$ is continuous.

1. If $f''(c) > 0$ then $x = c$ is: MINIMUM
2. If $f''(c) < 0$ then $x = c$ is: MAXIMUM
3. If $f''(c) = 0$ then the test is inconclusive. ⇝ Try first deriv. test.

Example: Using The Second Derivative Test

Find and classify the extrema of $f(x) = 3x^5 - 5x^3$.

Find the critical points.

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1) = 0$$

$$\Rightarrow x = -1, 0, 1$$

Apply second deriv test to each critical point.

$$f''(x) = 60x^3 - 30x = 30x(2x^2-1)$$

$$f''(-1) = 30(-1)(2(-1)^2-1) = -30$$

we calculate:

$$f''(-1) = -30 < 0 \Rightarrow x = -1 \text{ is a max}$$

$$f''(0) = 0 \longrightarrow \text{No conclusion.}$$

$$f''(1) = 30 > 0 \Rightarrow x = 1 \text{ is a min}$$

we apply first derivative test to values near $x = 0$:

$$f\left(\tfrac{1}{10}\right) = 15\left(\tfrac{1}{10}\right)^2\left(\left(\tfrac{1}{10}\right)^2-1\right) < 0$$

$$f\left(-\tfrac{1}{10}\right) = 15\left(-\tfrac{1}{10}\right)^2\left(\left(-\tfrac{1}{10}\right)^2-1\right) < 0$$

By the first deriv test, $x=0$ is neither max nor min.

Example: A Curve Sketch!

Sketch the graph of $f(x) = x^3 - 3x$.

- Find the x and y intercepts.
- Find the critical points of $f(x)$.
- Make a sign chart for $f'(x)$.
- Classify the critical points.

✓ y-intercept: $y = f(0) = 0$

✓ x-intercept: $x^3 - 3x = 0 \iff x(x^2-3) = 0$

$\iff x = -\sqrt{3}, 0, \sqrt{3}$

Critical points:

$f'(x) = 3x^2 - 3 = 3(x^2-1) = 0 \implies x = -1$ or $x = 1$ ✓ ✓

Sign Chart

Interval	Test Point	Sign of $f'(x)$	Conclusion
$(-\infty, -1)$	$x = -10$	$+$	$x = -1$ max ✓
$(-1, 1)$	$x = 0$	$-$	$x = 1$ min ✓
$(1, \infty)$	$x = 10$	$+$	

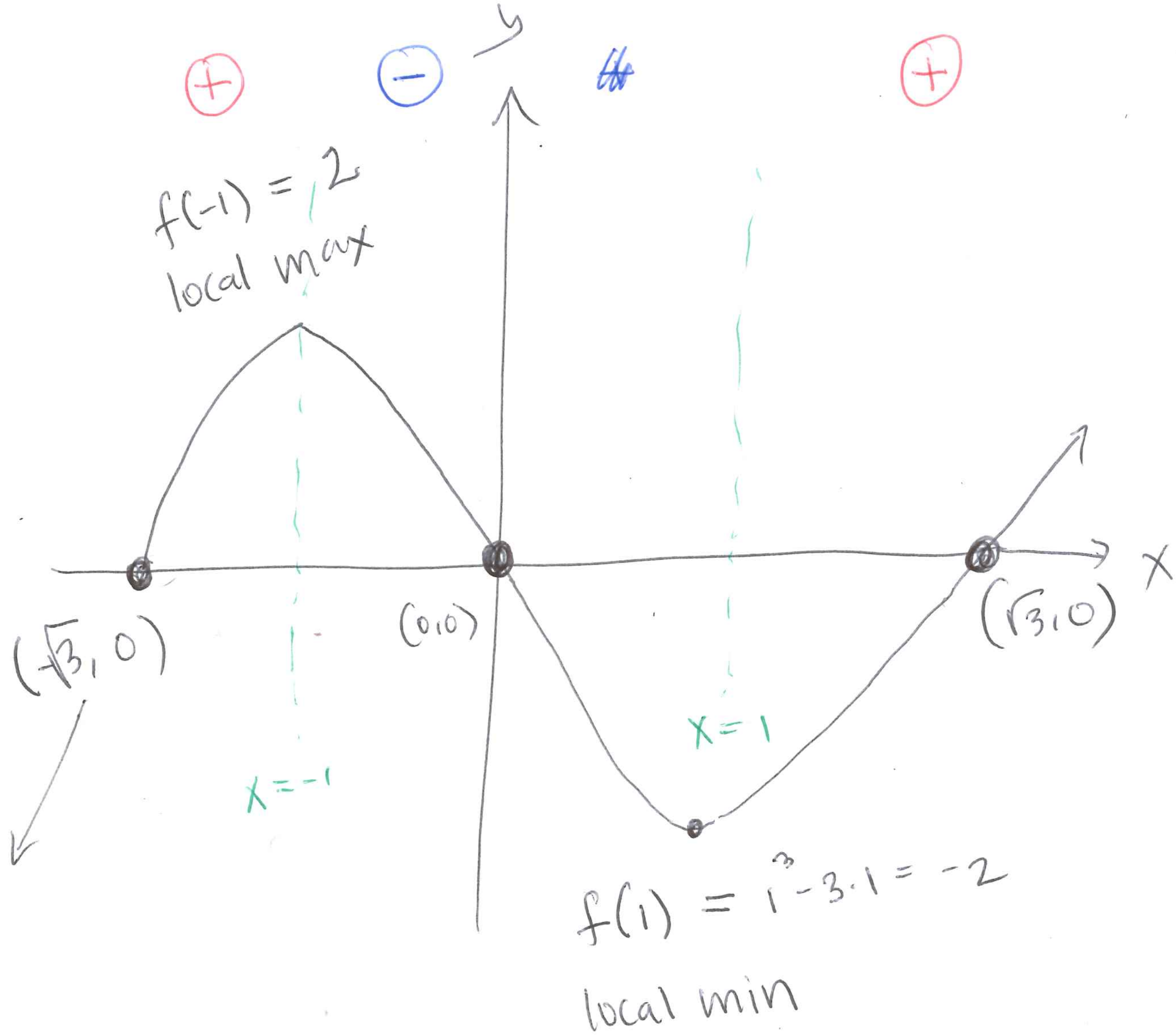
$f'(-10) = 3((-10)^2 - 1) = 3 \cdot 99 = 297 > 0$

$f'(10) = 3((10)^2 - 1) = 3 \cdot 99 > 0$

$f'(0) = 3(0^2 - 1) = -3 < 0$

Summary of Week 7

- Increasing and decreasing
- Concave up and down
- Second derivative test
- Sign charts



+
−
+
y
x
f(-1) = 2
local max
(-√3, 0)
(0,0)
(√3,0)
x = -1
x = 1
f(1) = 1³ - 3·1 = -2
local min