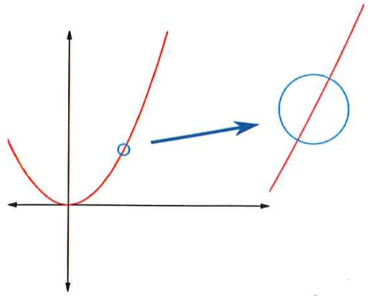
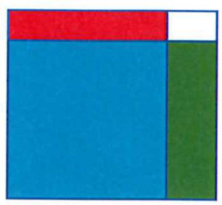
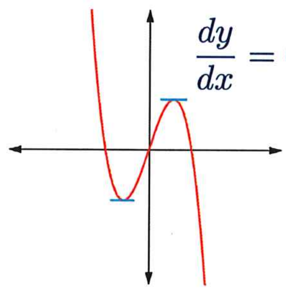
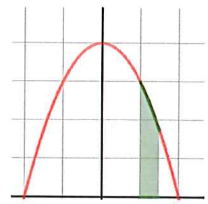


Week 6

MAT A29
Calculus I for the Life Sciences







Notes

The Lighthouse

Question (OS §4.1 Q10)

A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach. If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?

- Introduce variables.
  $x =$ distance from light on shore to P in miles, $t =$ time in minutes
  $\theta =$ angle of light from LP measured clockwise in radians
- State the known information

$$x = 2 \qquad \frac{d\theta}{dt} = \frac{10 \text{ revolutions}}{\text{minute}} = 10 \times 2\pi = 20\pi \text{ radians per minute}$$

- State the unknown.

$$\frac{dx}{dt} = \text{rate that the beam moves across the beach in miles per minute}$$

we don't have cos θ and sin θ! oh no!
However we know Opp and Adj!

Notes

By Pythagoras:

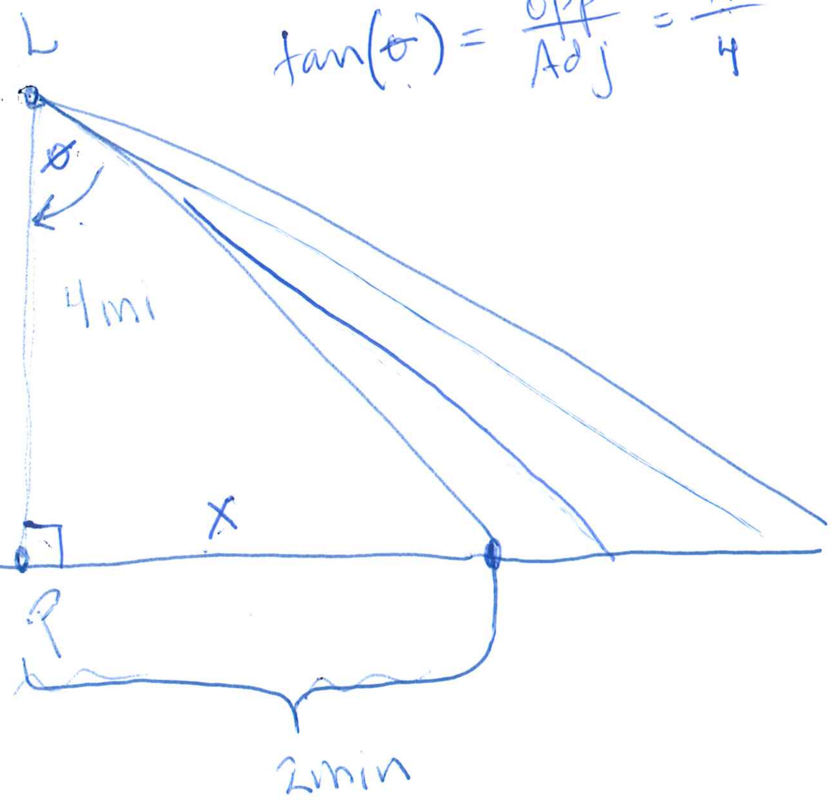
$$Hyp = \sqrt{Opp^2 + Adj^2} = \sqrt{2^2 + 4^2} = \sqrt{20}$$

We get:

$$\sin(\theta) = \frac{Opp}{Hyp} = \frac{2}{\sqrt{20}}$$

$$\cos(\theta) = \frac{Adj}{Hyp} = \frac{4}{\sqrt{20}}$$

$$\tan(\theta) = \frac{Opp}{Adj} = \frac{x}{4}$$



(5) Differentiate

$$\frac{d}{dt}\left[x^2+y^2\right]=\frac{d}{dt}\left[10^2\right]$$

$$\Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

(6) Apply known info.

$$2\cdot 5\cdot\frac{dx}{dt}+2y(-2)=0$$

↑ Unknown

Want ↗

Solve for wants:

$$2\cdot 5\frac{dx}{dt}=-2y(-2)$$

$$=-2(5\sqrt{3})(-2)$$

⇓

$$\frac{dx}{dt}=\frac{-2(5\sqrt{3})(-2)}{2\cdot 5}$$

(7) Solve for new unknown.

$$x^2+y^2=10^2\Rightarrow 5^2+y^2=10^2$$

$$\Rightarrow y^2=75$$

$$\Rightarrow y=\pm 5\sqrt{3}$$

We take $y=5\sqrt{3}$ because we want the ladder to be above ground.

(8) Solve!

$$\frac{dx}{dt}=\frac{-2(5\sqrt{3})(-2)}{2\cdot 5}=2\sqrt{3}\text{ ft/sec.}$$

:)

The Lighthouse

Question (OS §4.1 Q10)

A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach. If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min $= 20\pi$ *, how fast does the beam of light move across the beach* $x = 2$ *mi away from the closest point on the beach?*

- Relate the variables.

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{x}{4} \iff \frac{\sin(\theta)}{\cos(\theta)} = \frac{x}{4} \iff \boxed{4\sin(\theta) = x\cos(\theta)}$$

- Differentiate both sides.

$$\frac{d}{dt}[4\sin(\theta)] = \frac{d}{dt}[x\cos(\theta)] \iff 4\cos(\theta)\frac{d\theta}{dt} = \frac{dx}{dt}\cos(\theta) - x\sin(\theta)\frac{d\theta}{dt}$$

Chain rule

- Apply the given information.

$$4\cos(\theta)\frac{d\theta}{dt} = \frac{dx}{dt}\cos(\theta) - x\sin(\theta)\frac{d\theta}{dt} \iff 4\cos(\theta)20\pi = \frac{dx}{dt}\cos(\theta) - 2\sin(\theta)20\pi$$

rate of revolution

distance along shore

cos θ and sin θ are still unknown

From Geometry, we get: $\sin\theta = \frac{2}{\sqrt{20}}$ $\cos\theta = \frac{4}{\sqrt{20}}$

Notes

$$\frac{d}{dt}[x\cos(\theta)] = \frac{d}{dt}[x]\cos(\theta) + x\frac{d}{dt}[\cos(\theta)]$$

$$= \frac{dx}{dt}\cos(\theta) + x\frac{d\cos(\theta)}{d\theta}\cdot\frac{d\theta}{dt}$$

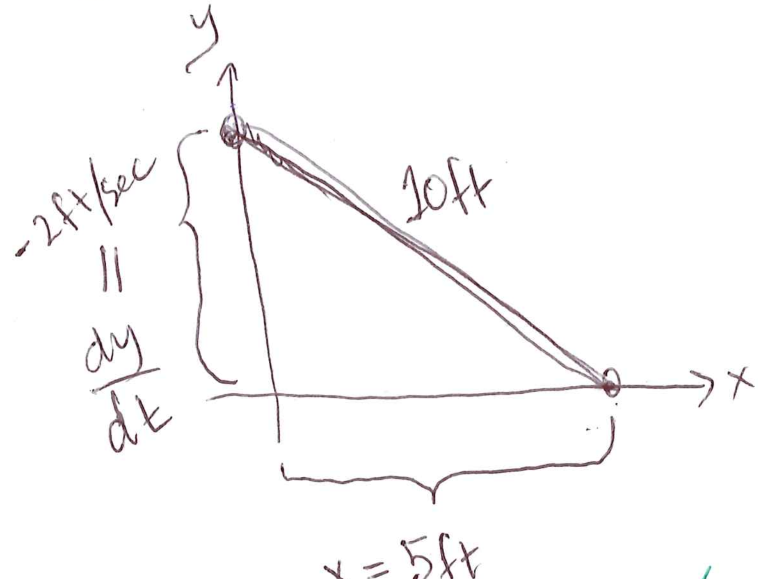
$$= \frac{dx}{dt}\cos(\theta) + x(-\sin\theta)\frac{d\theta}{dt}$$

$$= \frac{dx}{dt}\cos(\theta) - x\sin\theta\frac{d\theta}{dt}$$

Sliding Ladder

A 10ft long ladder leans against a wall.
Its top slides down the wall at 2ft/sec.
How fast is its bottom moving when
the bottom is 5 ft from the wall?

① Draw a Diagram



② Introduce notation

Let x be the distance from bottom of ladder to ~~floor~~ wall in ft.

y be the distance from top of ladder to floor in feet.

t be time in sec.

③ State known info./unknown info.

We know: $\frac{dy}{dt} = -2$ and $x = 5$.

We want to know: $\frac{dx}{dt} = ?$

④ Relate variables.

We know $x^2 + y^2 = 10^2$ by Pythagoras.

The Lighthouse

Question (OS §4.1 Q39)

A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach. If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach $x = 2$ mi away from the closest point on the beach?

- Solve for the new unknowns.
  We need some geometry! If $PX = 2$ and $LP = 4$ then Pythagoras gives us: $LX = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. The trigonometric ratios give us:

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}} \quad \sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$$

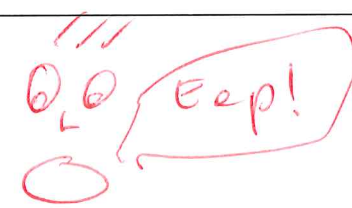
  Putting this together gives:

$$4\cos(\theta)20\pi = \frac{dx}{dt}\cos(\theta) - 2\sin(\theta)20\pi \iff \left(4\frac{2}{\sqrt{5}}20\pi\right) = \frac{dx}{dt}\left(\frac{2}{\sqrt{5}}\right) - \left(2\frac{1}{\sqrt{5}}20\pi\right)$$

- Solve the problem! $\frac{dx}{dt} = 100\pi$ mi / min

Notes

$$\frac{dx}{dt} = \frac{4\frac{2}{\sqrt{5}}20\pi + 2\frac{1}{\sqrt{5}}20\pi}{2/\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$$



$$= \frac{4\cdot 2\cdot 20\pi + 2\cdot 20\pi}{2} = \frac{160\pi + 40\pi}{2}$$

$$= \frac{200\pi}{2} = 100\pi$$

1. Draw diagram
2. Introduce notation
3. State knowns/unknowns
4. Relate variables
5. Differentiate
6. Apply known info
7. Solve new unknowns.

Big and Small

We often care about the biggest or smallest values of a function.

- Best rated restaurent
- Lowest cost of housing
- Shortest commute to a destination
- Longest possible nap
- Best grade possible

All of these situations are about maximizing or minimizing some quantity.

Notes

Absolute Extrema

Definition

The maximum or minimum values of a function $f(x)$ are called its extrema.
An absolute maximum $f(c)$ has the property that:

"global maximum" $$f(c) \geq f(x)$$

for all x in the domain.

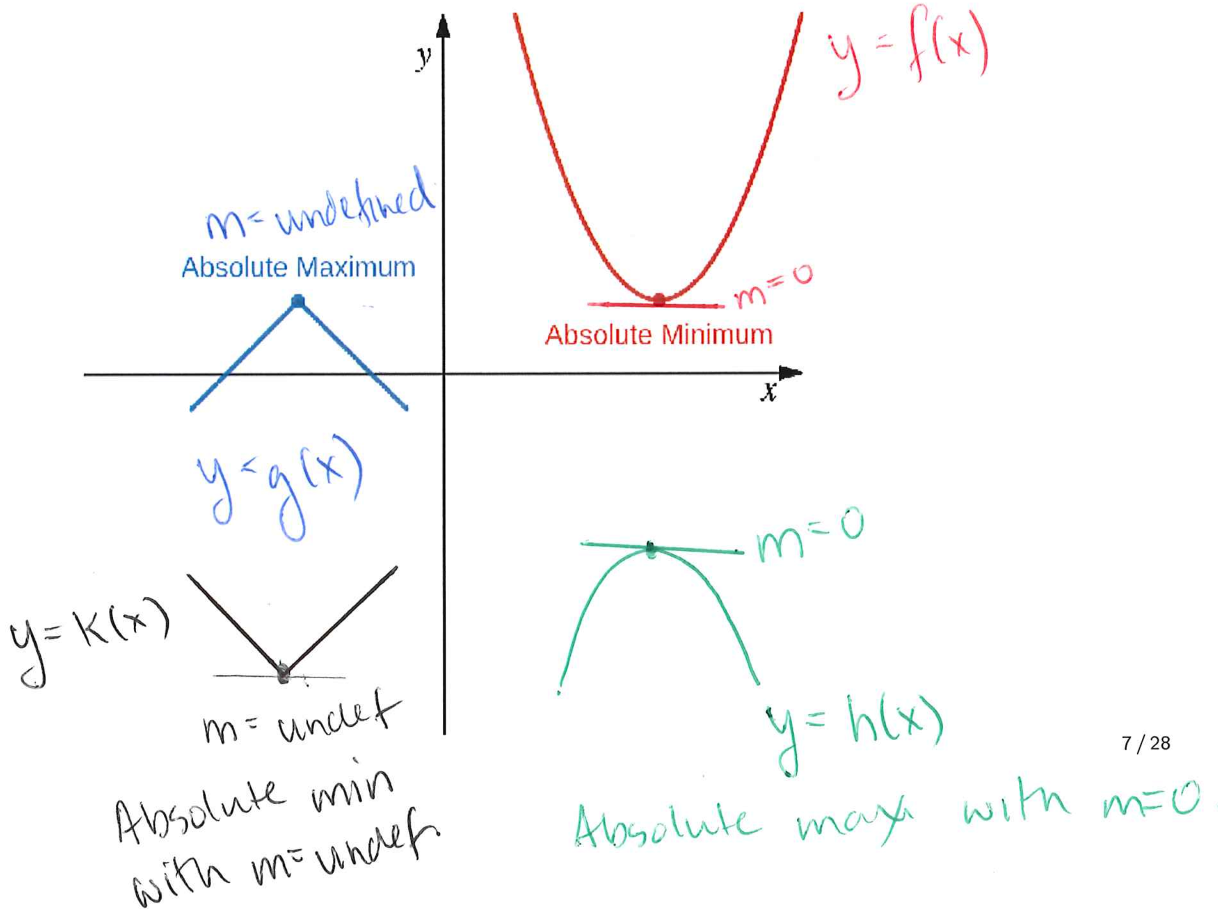
An absolute minimum $f(c)$ has the property that:

"global minimum" $$f(c) \leq f(x)$$

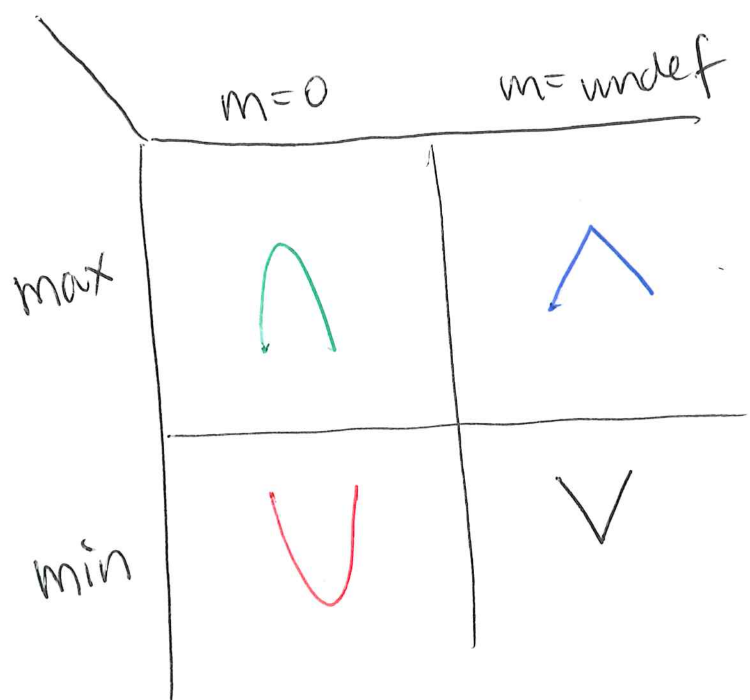
for all x in the domain.

Notes

Absolute Extrema



Notes



m = 0 and m = undef are related to extrema.

Local Extrema

Definition

A <u>local maximum</u> $f(c)$ has the property that:

"relative max"

$$f(c) \geq f(x)$$

for all x near c on both sides.

An <u>local minimum</u> $f(c)$ has the property that:

"relative min"

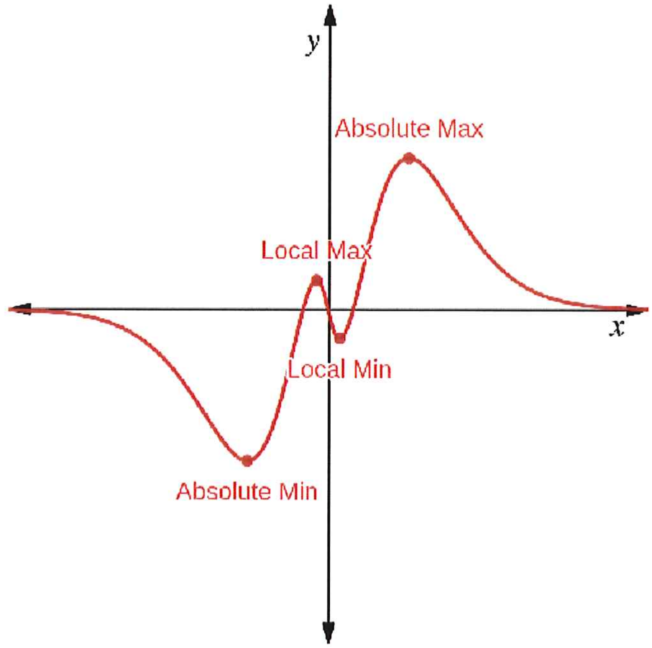
$$f(c) \leq f(x)$$

for all x near c on both sides.

Tricky: Absolute $\Rightarrow$ Local
If $f(c)$ is bigger than all values, then it is bigger than those near by.

Notes

Local Extrema



https://www.desmos.com/calculator/guxhpafzqi

Notes

Find the Maxmimum

Question (Think-Pair-Share (3 min))

Find the absolute maximum of $y = \dfrac{2}{x^2+9}$ *and argue that it has no absolute minimum.*

- Consider the function as a composition.

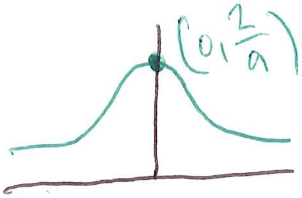
$$x \longrightarrow x^2 \longrightarrow x^2+9 \longrightarrow \frac{2}{x^2+9}$$

- We know that $x^2 \geq 0$ and $x^2 = 0$ only when $x = 0$.
- This gives us that $x^2 + 9 \geq 9$ and $x^2 + 9 = 9$ when $x = 0$.
- $x^2 + 9 \geq 9$ implies $\frac{2}{x^2+9} \leq \frac{2}{9}$ and equality occurs when $x = 0$.

Therefore, $y = \frac{2}{x^2+9} \leq \frac{2}{9}$ and $x = 0$ is an absolute maximum.
The function has no absolute minimum because $0 < \frac{2}{x^2+9}$ is never equal to zero.
The graph will get very close to $y = 0$ but it will never attain that value.

Notes

14:21



The range is $(0, \frac{2}{9}]$.

We can think of $y = \frac{2}{x^2+9}$ as a composition:

$$x \to x^2 \to x^2+9 \to \frac{2}{x^2+9}$$

The range of x^2 is $[0, \infty)$.

The range of x^2+9 is $[9, \infty)$.

The range of $\frac{2}{x^2+9}$ is $(0, \frac{2}{9}]$

no abs. min. (points to "(")

an absolute max (points to "]")

Find the Maxmimum

Question

A band knows that $100 - N$ people will attend their concert if they charge $2N\$$ to attend. How much should they charge to maximize their profit?

- Introduce a function.

$$P(N) = (\text{“attenders”})(\text{“profit per attender”}) = (100 - N)2N = 2N(100 - N)$$

- Analyze the resulting function.

$$P(N) = 200N - 2N^2 = -2N^2 + 200N \quad \leftarrow \text{Leading coefficient negative.}$$

- The resulting parabola opens downwards, and has its highest point at:

$$\underbrace{N = (0 + 100)/2}_{\text{magic from highschool}} = 50 \Longrightarrow P(50) = 2 \cdot 50(100 - 50) = 5000$$

Therefore, the band should charge $2N\$ = 100\$$ to attend their concert.
This solution used some magic from highschool. *Do we believe it?*

Notes

Notice: $N=0 \Rightarrow$ 100 people attend.

~~$N=50$~~ $N=100$ $\Rightarrow$ 0 people attend.

The possible values of N are $[0, 100]$.

We introduce the revenue function:

$$R(N) = (\text{people})(\text{cost of ticket})$$
$$= (100-N)(2N) = -2N^2 + 200N$$

This special function (a parabola) has a formula for where it achieves its max. (The vertex $N = \frac{0+100}{2}$)

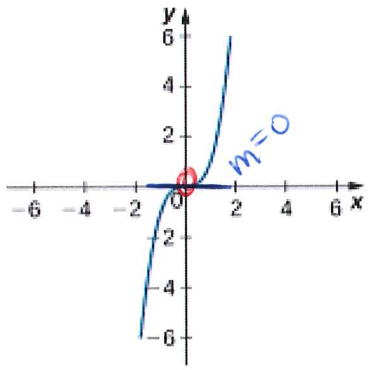
$R'(N)=0 \Leftrightarrow -4N+200=0 \Leftrightarrow N=50 \leftarrow$ 100\$ maximize revenue.

14:39

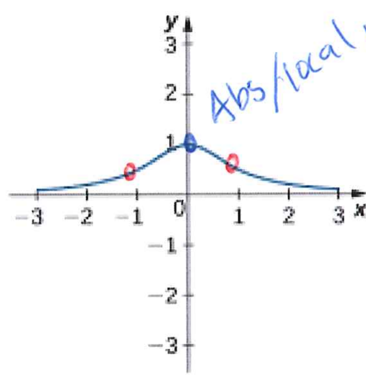
Describing Local and Absolute Extrema

Question (Think-Pair-Share 3 min)

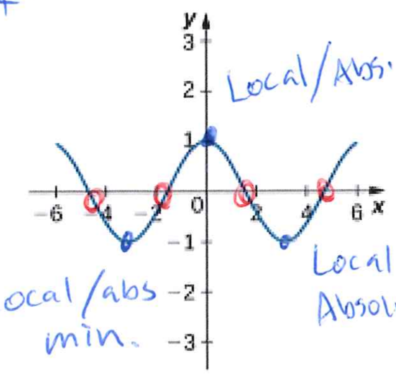
Describe the graphs shown below using the words "local, absolute, maximum, minimum".



$f(x) = x^3$ on $(-\infty, \infty)$



$f(x) = \frac{1}{x^2+1}$ on $(-\infty, \infty)$



$f(x) = \cos(x)$ on $(-\infty, \infty)$

OpenStax Fig 4.13

x=0 is not an extrema.

Attains abs. max but no abs. min.

Attains both its absolute max and min.

Points of inflection: the graph "switches sides of tangent line"

Notes

When thinking about extrema, the domain is very important!

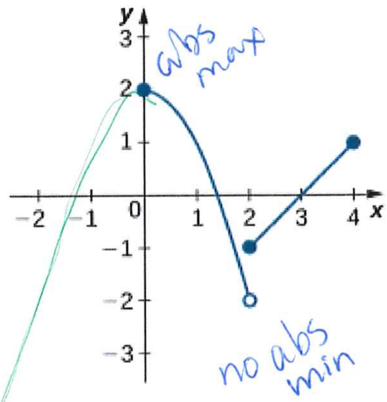
Changing the domain radically changes the behaviour of extrema.

Notes

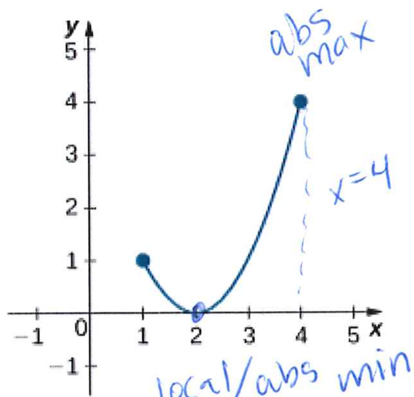
Describing Local and Absolute Extrema

Question

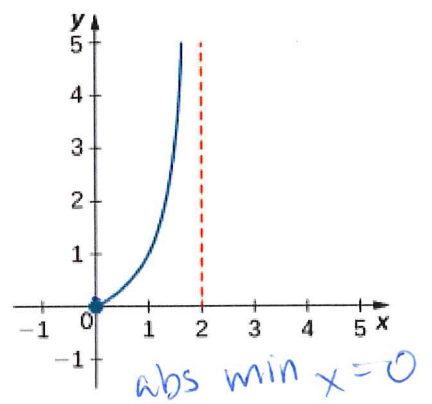
Describe the graphs shown below using the words "local, absolute, maximum, minimum".



$f(x) = \begin{cases} 2 - x^2 & 0 \le x < 2 \\ x - 3 & 2 \le x \le 4 \end{cases}$



$f(x) = (x - 2)^2$ on $[1, 4]$



$f(x) = \frac{x}{2-x}$ on $[0, 2)$

OpenStax Fig 4.13

$f(x) = \frac{x}{2-x}$ on $(-\infty, 2)$

Notes

$f(x)$ on $[-10, 4]$ will have an absolute min at $x = -10$.

Create.

$f(x) = (x-2)^2$ on $(-\infty, 2) \cup (2, \infty)$

This has no max nor min.

This no longer has an absolute min.

Destroy

The Extreme Value Theorem

Definition

A function is <u>continuous</u> at $x = c$ if:

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)$$

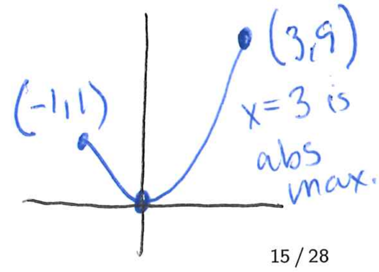
f cont. on [a,b] if it is cont. at x = c for all x ∈ [a,b]

Theorem

If $f(x)$ is a continuous function defined on $[a, b]$ then $f(x)$ has an absolute maximum and minimum.

Ex: $f(x) = x^2$ on $[-1, 3]$

The function f(x) is continuous because of the limit laws. It has abs. min. f(0) = 0 and abs. max. f(3) = 9.



x = 0 is abs min

Notes

Non-Ex: $f(x) = x^2$ on $[-1, 0) \cup (0, 3)$

This function has NO extrema on this domain.

Why Do We Need $[a, b]$ and Continuous?

Question (Discuss (5 min))

Describe the function below using the words: "local, absolute, maximum, minimum".

$$y = \begin{cases} x^2 & -1 < x < 0 \\ \frac{1}{42} & x = 0 \\ x^2 & 0 < x < 1 \end{cases}$$

Notes

Critical Points

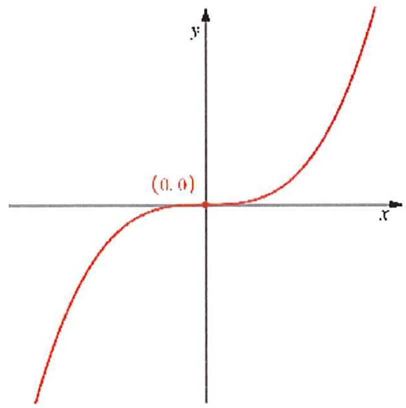
Definition

If $f'(c)$ is zero or undefined, then $x = c$ is a critical point of $f(x)$.

Theorem

If $f(x)$ has a local extrema at $x = c$ then $x = c$ is a critical point of $f(x)$.

Warning: Not all critical points are extrema. For example, $y = x^3$ at $x = 0$.
We have that $\frac{dy}{dx} = 3x^2 = 0$ when $x = 0$ but $(0, 0)$ is neither a max nor a min.



Notes

Locating Maxima and Minima

Question (OpenStax Ex. 4.13a)

Locate the maximum and minimum of $f(x) = -x^2 + 3x - 2$ *over* $[1, 3]$.

- Evaluate $f(x)$ at the endpoints.

$$f(1) = -1 + 3 - 2 = 0 \qquad f(3) = -9 + 9 - 2 = -2$$

- Determine any critical points.

$$f'(x) = 0 \Longrightarrow f'(x) = -2x + 3 = 0 \Longrightarrow x = \frac{3}{2}$$

- Evaluate at any critical points in $[1, 3]$.

$$f\left(\frac{3}{2}\right) = \frac{1}{4}$$

- Compare values to determine maximum and minimum.
  We conclude that $f(3) = -2$ is the absolute minimum and
  $f\left(\frac{3}{2}\right) = \frac{1}{4}$ is the absolute maximum.

Notes

Locating Maxima and Minima

Question (OpenStax Ex. 4.13a)

What happens to the question if the domain changes to $[2, 3]$*?*
Locate the maximum and minimum of $f(x) = -x^2 + 3x - 2$ *over* $[2, 3]$.

- Evaluate $f(x)$ at the endpoints.

$$f(2) = -4 + 6 - 2 = 0 \qquad f(3) = -9 + 9 - 2 = -2$$

- Determine any critical points.

$$f'(x) = 0 \Longrightarrow f'(x) = -2x + 3 = 0 \Longrightarrow x = \frac{3}{2}$$

- Evaluate at any critical points in $[2, 3]$.

There are no critical points in [2,3]!

- Compare values to determine maximum and minimum.
  We conclude that $f(3) = -2$ is the absolute minimum and
  $f(2) = 0$ is the absolute maximum.

Notes

The First Derivative Test (Sneak Peak to Week 7)

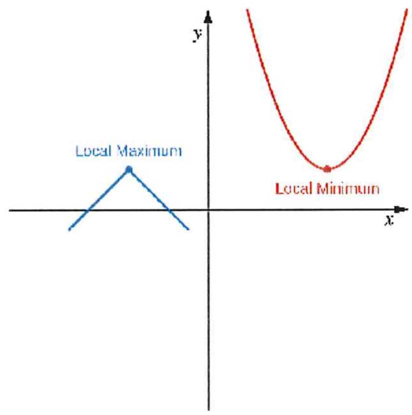
Theorem (OpenStax p.392)

Suppose $x = c$ is a critical point of $f(x)$.

If $f'(x)$ changes sign from $(+)$ when $x < c$ to $(-)$ when $x > c$ then:
$x = c$ is a local maximum.

If $f'(x)$ changes sign from $(-)$ when $x < c$ to $(+)$ when $x > c$ then:
$x = c$ is a local minimum.

If there is no change of sign, then $x = c$ is neither a max nor a min.



Notes

Using the First Derivative Test

Question

Use the first derivative test to classify the critical points of $y = (x^2 - 1)^3$.

- Determine the critical points.

$$f'(x) = \frac{d(x^2-1)^3}{d(x^2-1)} \frac{d(x^2-1)}{dx} = 3(x^2-1)^2(2x) = 0 \Longrightarrow x = -1,\ 0,\ 1$$

  Aha! $3(x^2-1)^2$ is always positive, and so the sign of $f'(x)$ is determined by $2x$.

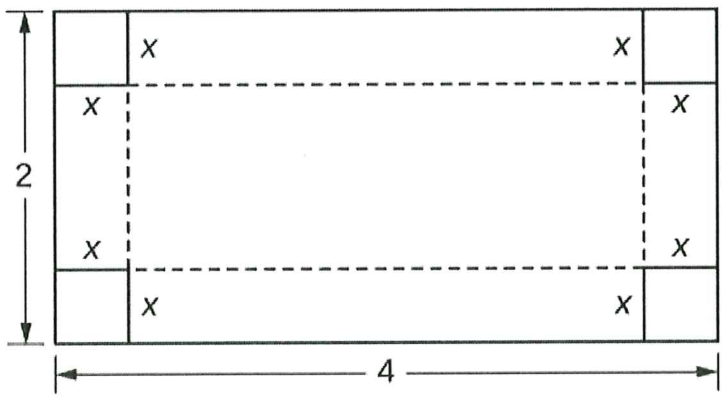
- Consider the spaces between these critical points.
  These are the intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$.
- Apply the first derivative test by calculating $f'(x)$ on each interval.
  $x = \pm 1$ are not max/min because $f'(x)$ has same sign on both sides of $x = c$.
  $x = 0$ is a minimum because $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$.

Notes

Making a Box

Question (OpenStax §4.7 Q 316)

You are constructing a cardboard box with the dimensions 2m by 4m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



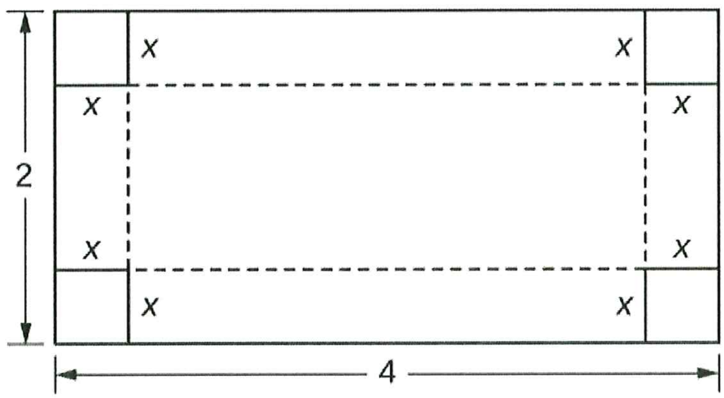
- Introduce variables.
  Let x be the sidelength of the corner squares in meters. Notice: $0 \leq x \leq 1$.

Notes

Making a Box

Question (OpenStax §4.7 Q 316)

You are constructing a cardboard box with the dimensions 2m by 4m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



- Introduce a function.
  $V(x) = (\text{length})(\text{width})(\text{height}) = (2-2x)(4-2x)x = 4(1-x)(2-x)x$

Notes

Making a Box

Question (OpenStax §4.7 Q 316 Mathematized)

Maximize $V(x) = 4(1-x)(2-x)x = 4x^3 - 12x^2 + 8x$ *subject to* $0 \le x \le 1$.

- Check the endpoints. $V(0) = V(1) = 0$. ← Probably not the maximum!
- Find critical points

  $$V'(x) = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2) = 0 \Longrightarrow x = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 3} = 1 \pm \frac{1}{\sqrt{3}}$$

  We can exclude the root $x = 1 + \frac{1}{\sqrt{3}} > 1$ because it does not satisfy $0 \le x \le 1$.Thus, $x = 1 - \frac{1}{\sqrt{3}} \approx 0.422$ is the only critical point we need to check.
- Apply the first derivative test on either side of the critical point $\frac{1}{10} < 1 - \frac{1}{\sqrt{3}} < \frac{1}{2}$.

  $$V'\left(\frac{1}{10}\right) = \frac{12}{100} - \frac{24}{10} + 8 > 0 \qquad V'\left(\frac{1}{2}\right) = \frac{12}{4} - \frac{24}{2} + 8 < 0$$

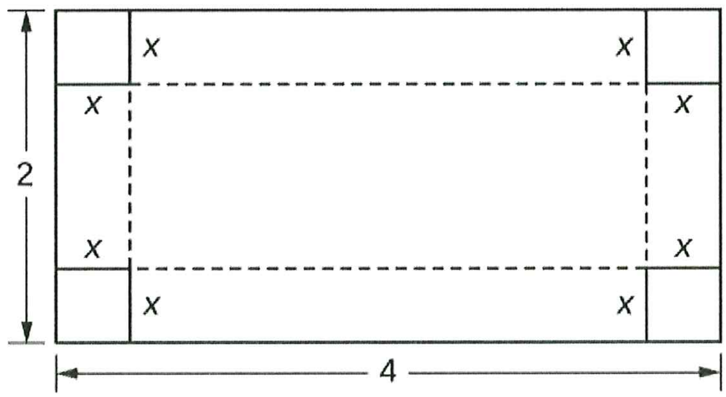
- Thus, $x = 1 - \frac{1}{\sqrt{3}}$ is a maximum.

Notes

Making a Box

Question (OpenStax §4.7 Q 316)

You are constructing a cardboard box with the dimensions 2m by 4m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



- Using $x = 1 - 1/\sqrt{3}$ we conclude that the box with the largest volume has length $2 - 2x = 2/\sqrt{3}$m, width $4 - 2x = 2 + 2/\sqrt{3}$m, and height $x = 1 - 1/\sqrt{3}$m.

Notes

Maximizing a Sum

Question (OpenStax §4.3 Q 318)

Find two integers greater than or equal to zero such that their sum is 10, and minimize and maximize the sum of their squares.

- Introduce notation.
  Let x and y be non-negative integers such that $x + y = 10$.
- State the quantity to maximize.

$$f(x, y) = x^2 + y^2$$

- Reduce the number of variables.

$$x + y = 10 \Rightarrow y = 10 - x \Rightarrow f(x) = x^2 + (10 - x)^2 = 2x^2 - 20x + 100$$

We have that $0 \leq x \leq 10$.

Notes

Maximizing a Sum

Question (OpenStax §4.3 Q 318)

Maximize $f(x) = x^2 + (10 - x)^2$ *when* $0 \leq x \leq 10$.

- Check the endpoints. $f(0) = f(10) = 100$.
- Find the critical points.

$$f'(x) = 2x + 2(10 - x)(-1) = 4x - 20 = 4(x - 5) \implies x = 5$$

- Apply the first derivative test.
  If $x < 5$ then $f'(x) = 4(x - 5) < 0$.
  If $x > 5$ then $f'(x) > 0$.
- Write a conclusion.
  Thus, $x = 5$ is a minimum!The two positive integers whose sum is 10 which maximize the sum of their squares are $(x, y) = (10, 0)$ or $(0, 10)$.

Notes

Summary of Week 6

- Absolute and local extrema
- First derivative test
- Importance of end points and continuinty

Notes