…nd Slope

…on (OpenStax Pg. 220)

…efine the <u>derivative</u> $f(x)$ at $x = a$ to be:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \text{“ } f \text{ prime at } a \text{ ”}$$

Definition (OpenStax Pg. 234)

Leibniz: $$\frac{dy}{dx}\bigg|_{x=a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{“ dee } y \text{ dee } x \text{ ”}$$ → SLOPE

We will use $\frac{d}{dx}[\cdot]$ as a way of writing “take the derivative”.

$$\frac{d}{dx}[\text{stuff}] = \text{derivative of stuff}$$

Notes

Missing Pages (on this hardcopy & Quercus)
$\#\{6, 9, 18, 28, 32-35\}$

Week → 4

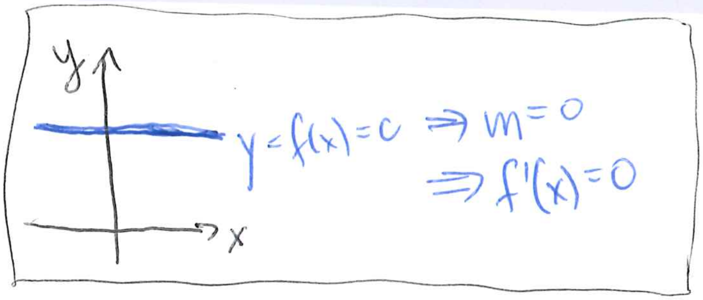
Pages 22 - onwards are here in hardcopy but missing on Quercus

Constants!

Theorem (OpenStax §3.3 Theorem 3.2)

$$\frac{d}{dx}[c] = 0$$

Alternatively, if $f(x) = c$ is constant and does not change, then $f'(x) = 0$.



The algebra/limits version.
We calculate:

$\boxed{\text{Let } f(x) = c \text{ for all } x.}$

This gives:

$$\frac{d}{dx}[c] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h}$$

$$= \lim_{h\to 0} 0 \quad (\text{for } h\neq 0)$$

Notes $= 0$

Composition of functions is a big deal!

$f(x) = \ldots$

we almost always get

$f(x) = g(h(\ldots k(x)))$

Notes

Powers

Theorem (OpenStax §3.3 Theorem 3.3)

If n is a number,

$$\boxed{\frac{d}{dx}[x^n] = nx^{n-1}}$$

(We consider just $n=3$.) Let $f(x)=x^3$. We get:

$$\frac{d}{dx}[x^3] = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$

$$= \lim_{h\to 0}\frac{(\cancel{x^3}+3x^2h+3xh^2+h^3)-\cancel{x^3}}{h}$$

$$= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h} = \text{"}\frac{0(\ldots)}{0}\text{"}$$

$$= \lim_{h\to 0} 3x^2+3xh+h^2 \ (\text{for } h\neq 0) = 3x^2 = \boxed{3x^{3-1}}$$

Notes

Aside: We expand

$$(x+h)^3 = (x+h)(x+h)(x+h)$$
$$= (x+h)(x^2+xh+hx+h^2)$$
$$= (x+h)(x^2+2xh+h^2)$$
$$= x^3+2x^2h+xh^2$$
$$+ hx^2+2xh^2+h^3$$
$$= x^3+3x^2h+3xh^2+h^3$$

$$\boxed{\text{"}nx^{n-1}h\text{"}}$$

Why do we write

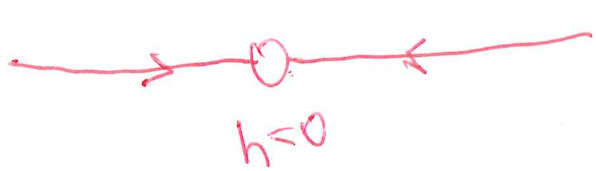
$$\lim_{h \to 0} \frac{h(\ldots)}{h}$$

$$= \lim_{h \to 0} (\ldots) \quad \text{for } h \neq 0$$

and then let $h = 0$?!

The big deal here is that cancelling h from top and bottom changes the function. The function changes because it becomes defined at $h=0$. (The domain changes to include $h=0$.)

The amazing fact about limits is they let us evaluate functions at points outside their domain.



Linearity

Discuss 2min.

Theorem (OpenStax §3.3 Theorem 3.4)

For any numbers a and b: $\boxed{\dfrac{d}{dx}[af(x)+bg(x)] = af'(x)+bg'(x).}$

Sum Rule ($a=b=1$)

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

Difference Rule ($a=1$ and $b=-1$)

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

Constant Multiple Rule (□)

$$\frac{d}{dx}[kf(x)] = kf'(x)$$

?

Take $a=k$.
$b=0$.

We calculate:

$$\frac{d}{dx}\left[af(x)+bg(x)\right] = \lim_{h\to 0}\frac{\left[af(x+h)+bg(x+h)\right]-\left[af(x)+bg(x)\right]}{h}$$

Notes

$$= \lim_{h\to 0}\left[\frac{a\left(f(x+h)-f(x)\right)}{h} + \frac{b\left(g(x+h)-g(x)\right)}{h}\right]$$

$$= a\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + b\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= af'(x)+bg'(x)$$

Add more time.

13:45

olynomials!

Activity (Micro-Assignment (5 min))

Compute the derivative of $f(x) = x^3 - 2x + 1$.
Either: use the limit definition, or carefully explain every step using the rules from this class.

Limit Defn

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[(x+h)^3 - 2(x+h) - 1\right] - \left[x^3 - 2x + 1\right]}{h}$$

$$= \ldots \quad (\text{Algebra})$$

$$= \lim_{h\to 0} (\ldots) \text{ for } h \neq 0.$$

$$= 3x^2 - 2.$$

Deriv Rules

$$\frac{d}{dx}\left[x^3 - 2x + 1\right]$$

$$= \frac{d}{dx}\left[x^3\right] - \frac{d}{dx}\left[2x\right] + \frac{d}{dx}\left[1\right] \quad \text{\# sum and diff.}$$

$$= 3x^2 - \frac{d}{dx}\left[2x\right] + 0 \quad \text{\# const and polynomials}$$

$$= 3x^2 - 2\frac{d}{dx}\left[x\right] \quad \text{\# const. multiple}$$

$$= 3x^2 - 2\cdot 1 = 3x^2 - 2.$$

Notes

$$\frac{d}{dx}\left[x^3\right] = 3x^{3-1} = 3x^2$$

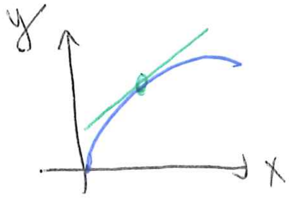
Using Derivatives

Definition (OpenStax Pg. 217)

The tangent slope to $y = f(x)$ is:

$$m_{tan} = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The tangent line to $y = f(x)$ at $x = a$ has slope m_{tan} and passes through $(a, f(a))$.



The tangent line has exactly the same slope as $y = f(x)$ at $x = a$, and passes through $(a, f(a))$.

Notes

Using Derivatives

Question

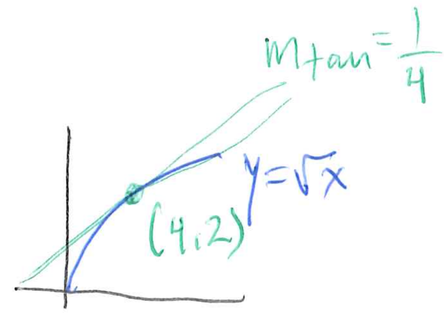
Find the tangent slope of $f(x) = \sqrt{x}$ at $a = 4$.

$$m_{tan} = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - \sqrt{4}}{x - 4} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x \to 4} \frac{\sqrt{x} - \sqrt{4}}{(\sqrt{x} + \sqrt{4})(\sqrt{x} - \sqrt{4})}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x} + \sqrt{4}} \quad (\text{for } x \neq 4)$$

$$= \frac{1}{\sqrt{4} + \sqrt{4}} = \frac{1}{2+2} = \frac{1}{4}$$



Notes

Using Derivatives

Question

Find the tangent line of $f(x) = \sqrt{x}$ at $a = 4$ assuming $m_{tan} = \frac{1}{4}$.

The point $(4, \sqrt{4}) = (4, 2)$ is on ~~the~~ both $y = \sqrt{x}$ and the tangent line. We use point-slope format and get:

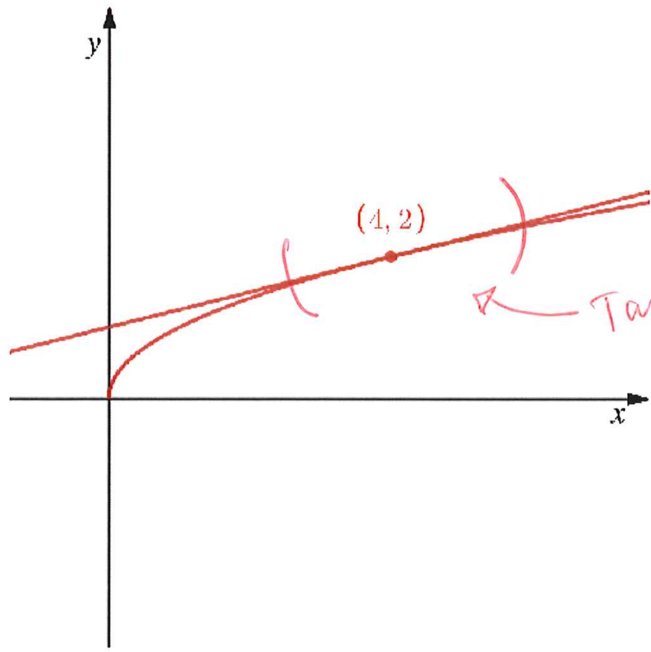
$$y - 2 = m_{tan}(x - 4)$$

$$y - 2 = \frac{1}{4}(x-4) = \frac{1}{4}x - 1$$

$$\iff y = \frac{1}{4}x + 1.$$

Notes

Using Derivatives



Tangent line is very close to $y = \sqrt{x}$

https://www.desmos.com/calculator/luadl4c0iy

Notes

Week 8 Sneak Peek!

Replace hard $\sqrt{x}$ by $\frac{1}{4}x+1$ easier.

Question

Use the tangent line $y = \frac{1}{4}x + 1$ of $y = \sqrt{x}$ at $a = 4$ to approximate $\sqrt{4.242}$.

What's the idea here? $4.242 \approx 4.000$.
Near the point of tangency, $\frac{1}{4}x + 1 \approx \sqrt{x}$.

$$\frac{1}{4}(4.242) + 1 = 2.0605 \qquad \sqrt{4.242} = 2.0596\ldots$$

That's a pretty good approximation!

Notes

Using Derivatives

Question

Where is the tangent line of $y = x^3 - 3x$ horizontal?

- Horizontal tangent $\iff m_{tan} = 0$
- $m_{tan} = \frac{dy}{dx} = \frac{d}{dx}[x^3 - 3x] = 3x^2 - 3 = 3(x^2-1) = 0$

$\iff 3(x-1)(x+1) = 0$

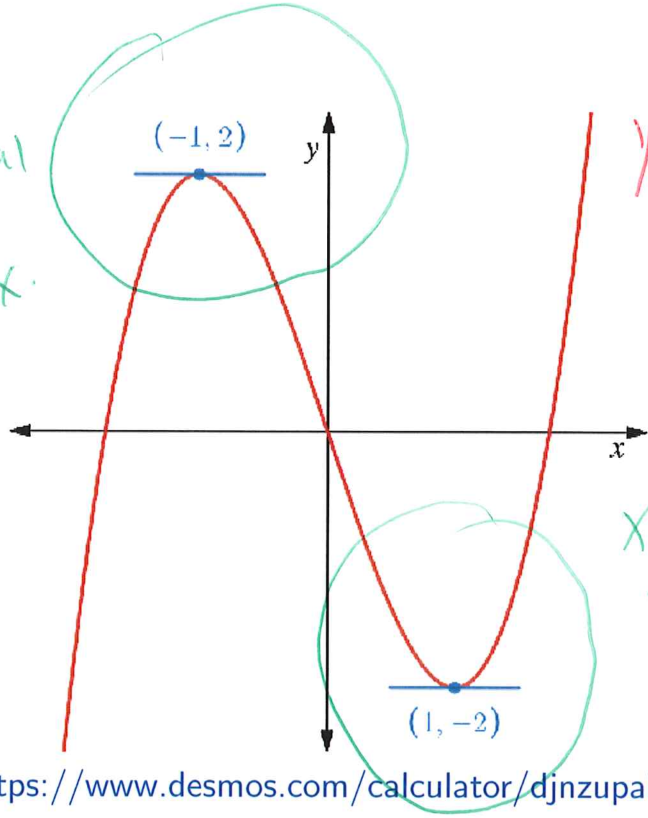
$\iff x = 1$ or $x = -1$.

The tangent line at ~~x=1~~ $x=-1$ or $x=1$ is horizontal.

Notes

Using Derivatives

$x=-1$ is a local maximum of $f(x)=x^3-3x$.



$y = x^3 - 3x$

$x=1$ is a local minimum of $f(x)=x^3-3x$

https://www.desmos.com/calculator/djnzupah2i

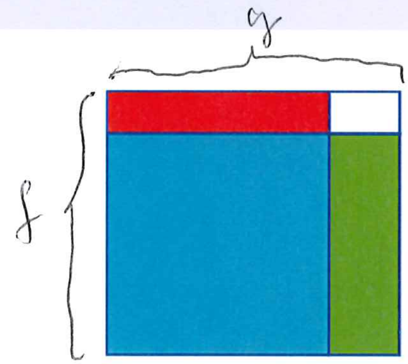
Local means "near the value".

Notes

Product Rule

Theorem (OpenStax §3.3 Pg. 253)

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$



$$\frac{d}{dx}\left[fg\right] = f'g + fg'$$

Notes

Add and subtract additional terms.

$$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{g(x+h)\left[f(x+h) - f(x)\right]}{h} + \frac{f(x)\left[g(x+h) - g(x)\right]}{h}\right]$$

$$= g(x+0)f'(x) + f(x)g'(x) = f'(x)g(x) + f(x)g'(x).$$

An Absolute Surprise

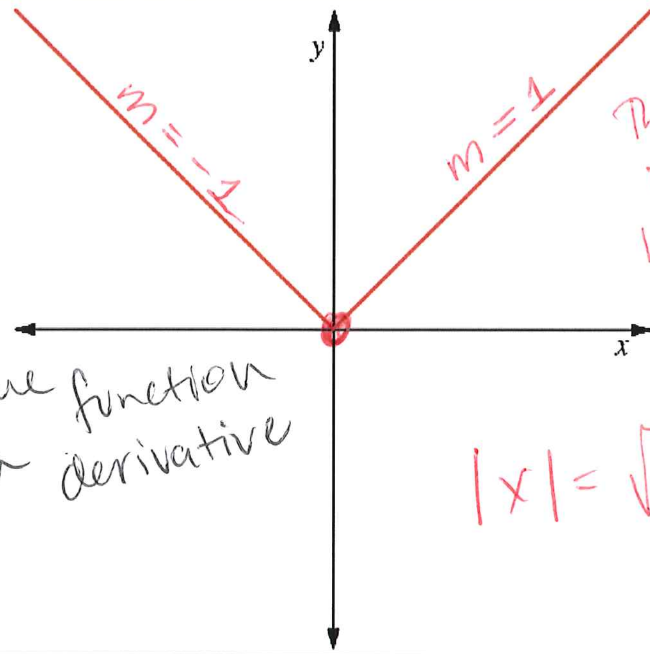
Definition

The absolute value function $f(x) = |x|$ is defined as:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$\pi = |-\pi| = -(-\pi)$

Why would anyone define this thing? It measures the distance from x to 0.



The distance from x to 0 must be positive: |x|.

The absolute value function does not have a derivative at $x=0$.

$|x| = \sqrt{x^2}$

Notes

An Absolute Surprise

Question

What's the slope of $y = |x|$ at $(x, y) = (0, 0)$?

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

To evaluate this limit, we need to calculate $\lim_{h\to 0^+} \frac{|h|}{h}$ ad $\lim_{h\to 0^-} \frac{|h|}{h}$. We calculate $\lim_{h\to 0^+} \frac{|h|}{h}$ first.

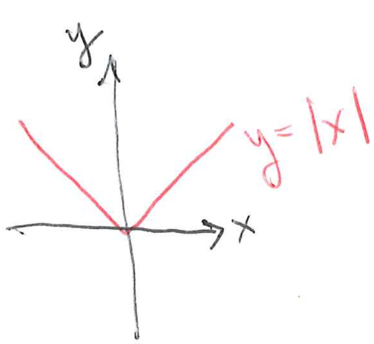
$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 \ (\text{for } h \neq 0) = \underline{1}.$$

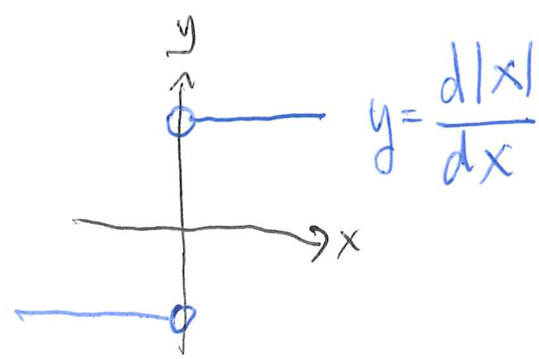
Notes

On the other side:

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 \ (\text{for } h\neq 0) = -\underline{1}.$$

This gives $\lim_{h\to 0^+} \neq \lim_{h\to 0^-}$ ad so $\lim_{h\to 0}$ does not exist.





On the TT1 Description
" Limits as Derivatives"

$$\lim_{h\to 0} \frac{e^h - 1}{h}$$

What is This the Derivative Of?

Question

The $\lim_{h\to 0} \frac{\sqrt{h^2 - h}}{h}$ *represents which of the following derivative calculations?*

1. $f'(0)$ *where* $f(x) = \sqrt{x^2 - x}$
2. $g'(x)$ *where* $g(x) = \sqrt{x^2}$
3. $h'(1)$ *where* $h(x) = x^2 - x$
4. $u'(-1)$ *where* $u(x) = x^2$

(5 min)

$$f'(x) = \boxed{\lim_{h\to 0}} \frac{\boxed{f(x+h)} - \boxed{f(x)}}{\boxed{h}}$$

Notes

$$\lim_{h\to 0} \frac{\sqrt{h^2-h}}{h} = \lim_{h\to 0} \frac{\sqrt{h^2-h} - 0}{h} = \lim_{h\to 0} \frac{\sqrt{h^2-h} - \sqrt{0^2-0}}{h}$$

$$= \lim_{h\to 0} \frac{\boxed{\sqrt{(0+h)^2 - (0+h)}} - \boxed{\sqrt{0^2-0}}}{h}$$

= " the derivative of $\sqrt{x^2 - x}$ at $x = 0$ "

= 1.

$$\frac{1+2}{3} = \frac{1}{3} + \frac{2}{3}$$

Quotient Rule

Theorem (OpenStax §3.6 Pg. 255)

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$ ← *The messiest derivative rule!*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}\left(\frac{g(x+h)g(x)}{g(x+h)g(x)}\right)$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)}{\cancel{g(x+h)}}\cancel{g(x+h)}g(x) - \frac{f(x)}{\cancel{g(x)}}g(x+h)\cancel{g(x)}}{h\,g(x+h)g(x)}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x+h)g(x)}$$

This cancellation is tricky.

Notes

$$= \lim_{h\to 0} \frac{\boxed{f(x+h)g(x) - f(x)g(x)} + \boxed{f(x)g(x) - g(x+h)f(x)}}{h\,g(x+h)g(x)}$$

Looks like f'

$$= \lim_{h\to 0}\left[\frac{1}{g(x+h)g(x)}\,\frac{f(x+h)g(x)-f(x)g(x)}{h}\right] +$$

Looks like g'

$$\left[\frac{1}{g(x+h)g(x)}\,\frac{f(x)g(x)-g(x+h)f(x)}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{1}{g(x)g(x+h)} \frac{f(x+h)g(x) - f(x)g(x)}{h} \right]$$

$$- \left[\frac{1}{g(x)g(x+h)} \frac{g(x+h)f(x) - g(x)f(x)}{h} \right]$$

$$= \lim_{h \to 0} \frac{1}{g(x)g(x+h)} g(x) \left[\frac{f(x+h) - f(x)}{h} \right]$$

$$- \frac{1}{g(x)g(x+h)} f(x) \left[\frac{g(x+h) - g(x)}{h} \right]$$

$$= \frac{1}{[g(x)]^2} \left[g(x) f'(x) \right] - \frac{1}{[g(x)]^2} \left[f(x) g'(x) \right]$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Yay! The Quotient Rule.

$$\lim_{h \to 0} \frac{1}{g(x)g(x+h)} = \frac{1}{g(x)g(x+0)} = \frac{1}{[g(x)]^2}$$

Question

Apply the quotient rule to $k(x) = \dfrac{5x^2}{4x+3}$.

We calculate:

$$k'(x) = \frac{\frac{d}{dx}\left[5x^2\right](4x+3) - 5x^2\frac{d}{dx}\left[4x+3\right]}{(4x+3)^2}$$

$$= \frac{10x(4x+3) - 5x^2(4)}{(4x+3)^2}$$

Notes

$$= \frac{40x^2 + 30x - 20x^2}{(4x+3)^2}$$

$$= \frac{20x^2 + 30x}{(4\cancel{x}+3)^2} = \frac{10x(2x+3)}{(4x+3)^2}$$

$$= \frac{20x^2+30x}{16x^2+24x+9}$$

Apply the Quotient Rule

(2 min)

Question

What is the derivative of $f(x) = \frac{x+4}{x+1}$ *at the point* $x = 2$?

1. $f'(x) = -3$
2. $f'(x) = -\frac{1}{3}$
3. $f'(x) = 2$
4. *f does not have a derivative at* $x = 2$.

$$f'(x) = \frac{\frac{d}{dx}[x+4](x+1) - (x+4)\frac{d}{dx}[x+1]}{(x+1)^2}$$

$$= \frac{1(x+1) - (x+4)1}{(x+1)^2} = \frac{-3}{(x+1)^2}$$

Notes

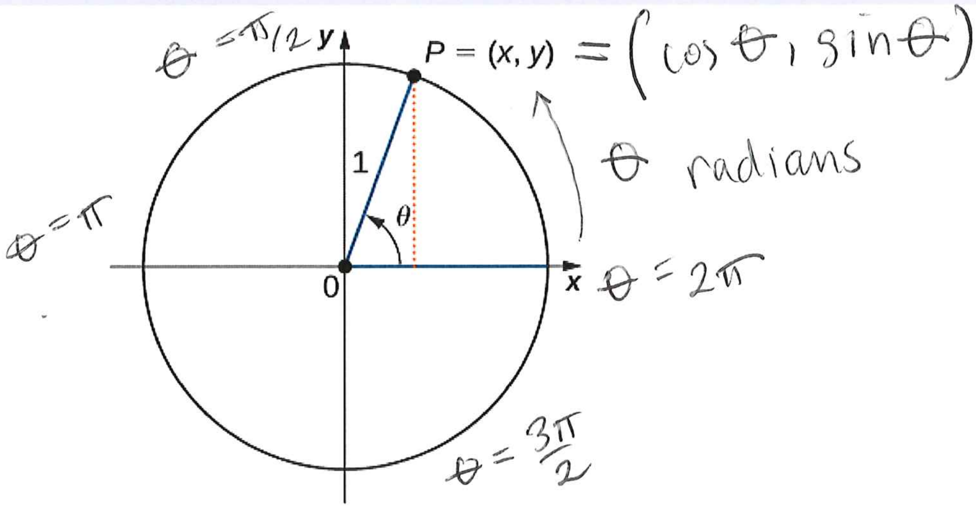
This gives:

$$f'(2) = \frac{-3}{(2+1)^2} = -\frac{3}{9} = -\frac{1}{3}$$

The Trigonometric Functions sin and cos

Question

How are $\sin(x)$ *and* $\cos(x)$ *related?*



OpenStax §1.3 Fig 1.31

$$(x, y) = (\cos(\theta), \sin(\theta)) \iff \boxed{\sin^2(\theta) + \cos^2(\theta) = 1}$$

Notes

Various other Trig Functions

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$ "cotangent"

$\sec\theta = \frac{1}{\cos\theta}$ "secant"

$\boxed{\csc\theta = \frac{1}{\sin\theta} \text{ "cosecant"}}$

These are all "just" ways of re-writing $\sin\theta$ and $\cos\theta$.

$$\frac{d}{dx}\left[\csc(x)\right] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right]$$

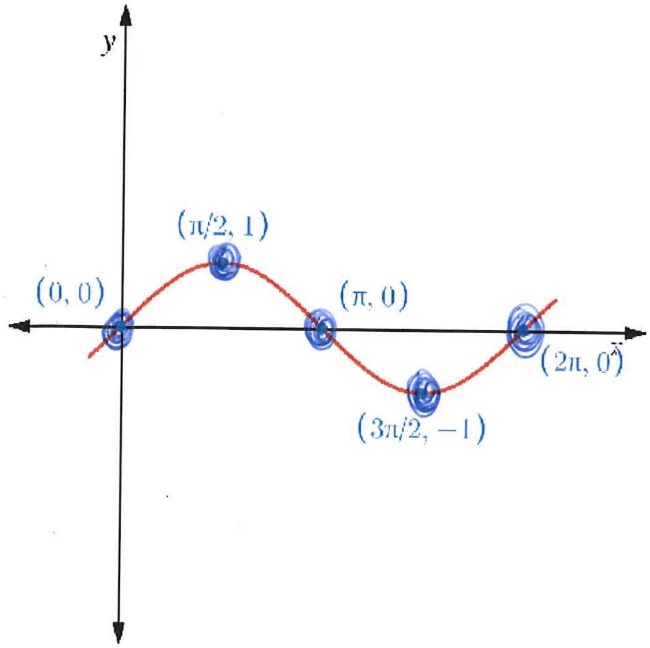
$$= \frac{\frac{d}{dx}[1]\sin(x) - 1\,\frac{d}{dx}[\sin(x)]}{[\sin(x)]^2}$$

$$= \frac{0 \cdot \sin(x) - 1 \cdot \cos(x)}{[\sin(x)]^2}$$

$$= -\frac{\cos(x)}{\sin^2(x)} = \frac{-\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)}$$

$$\cancel{= (-1)\cot(x)\sec(x)}$$

Tangents to the sin Curve

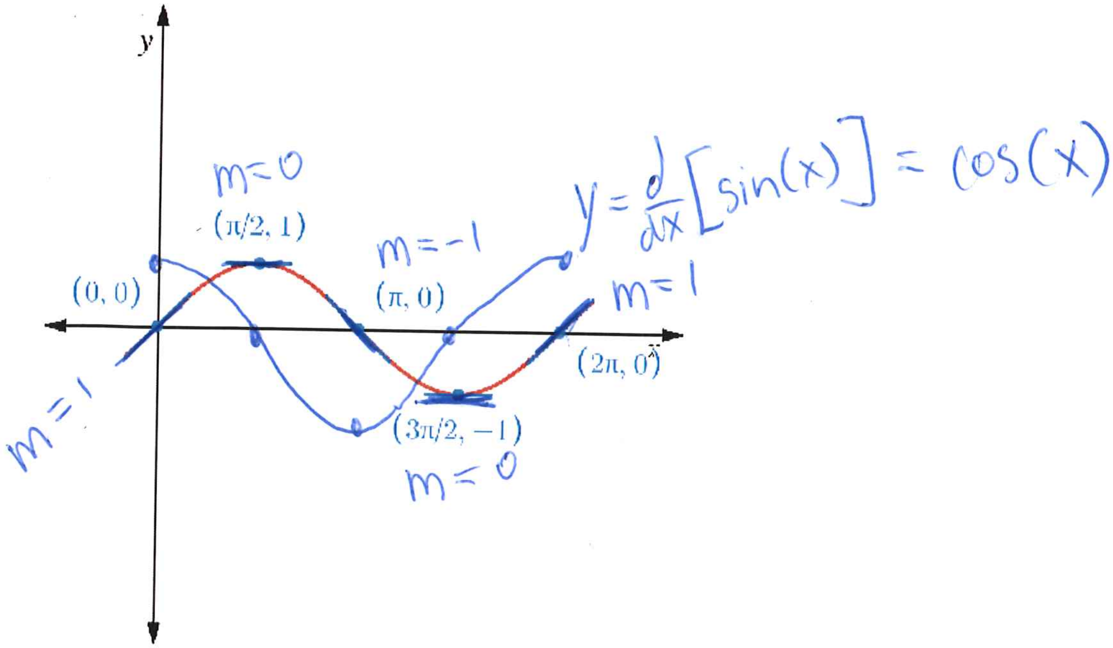


$\sin(0) = 0$
$\sin\left(\frac{\pi}{2}\right) = 1$
$\sin(\pi) = 0$
$\sin\left(\frac{3\pi}{2}\right) = -1$
$\sin(2\pi) = 0.$

$y = \sin(x)$ with important points highlighted

Notes

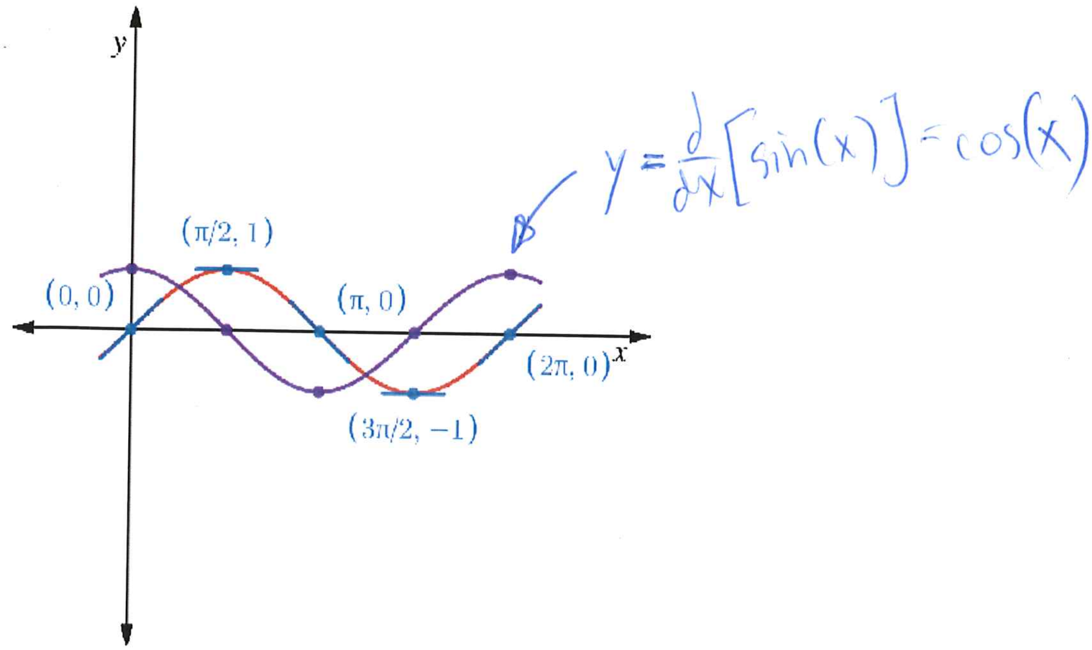
Tangents to the sin Curve



$y = \sin(x)$ with tangent lines at important points

Notes

Tangents to the sin Curve



$y = \sin(x)$ with derivative curve

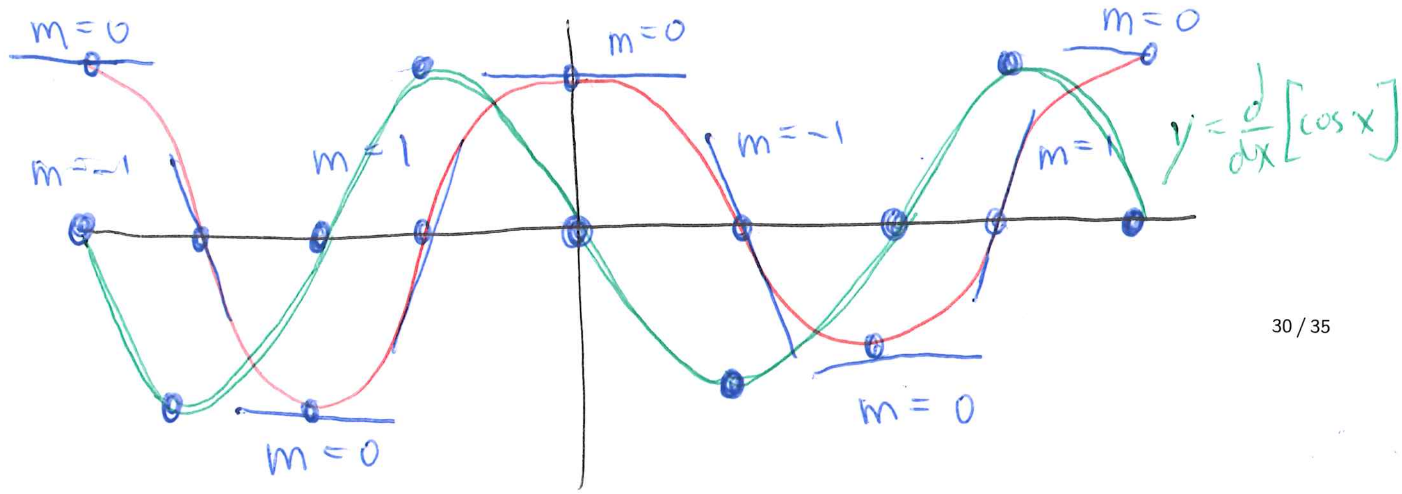
Notes

Tangents to the sin Curve

Theorem

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

The Same Drawing for $y = \cos(x)$



Notes

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$\cos(0) = 1$

$\cos(\pm\frac{\pi}{2}) = 0$

$\cos(\pm\pi) = -1$

$\cos(\pm\frac{3\pi}{2}) = 0$

$\cos(\pm 2\pi) = 1$

What Can the Derivative Rules Do?

Question

Which of the following derivatives can we NOT compute?
(Assuming the derivative rules we have so far.)

1. $f(x) = x^3 + 2x + 1$ ✓
2. $g(x) = \sqrt{2x+3}$ → We need more tools.
3. $h(x) = \frac{x}{x^2+1}$ ✓
4. $i(x) = x\sin(x)$ ✓

Notes