What Can the Derivative Rules Do?

Question

Which of the following derivatives can we NOT compute? (Assuming the derivative rules we have so far.)

1. $f(x) = x^3 + 2x + 1$
2. $g(x) = \sqrt{2x+3}$ ← Need chain rule
3. $h(x) = \frac{x}{x^2+1}$
4. $i(x) = x\sin(x)$

Notes

Week-~~6~~ 5
- Not covered
- Page missing (hardcopy) 2, 4
- All pages missing from Quercus.

ne Chain Rule

Theorem (OpenStax p.288)

If $y = f(u)$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

"chain"

The quantities $\frac{dy}{du}$ ad $\frac{du}{dx}$ are NOT fractions.

y u u x

If y is a quantity that depends on u, and u is a quantity that depends on x, then it must be that y depends on x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

They are limits of fractions:

$$\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}$$

Notes

An Example with Money

Example

Suppose that UTSC shirts cost 10$ / shirt.
If you buy five shirts, how much will this purchase cost? 50$ It will cost fifty bucks!

- Introduce variables: S = shirts, M = money, t = time.
- State the known information:

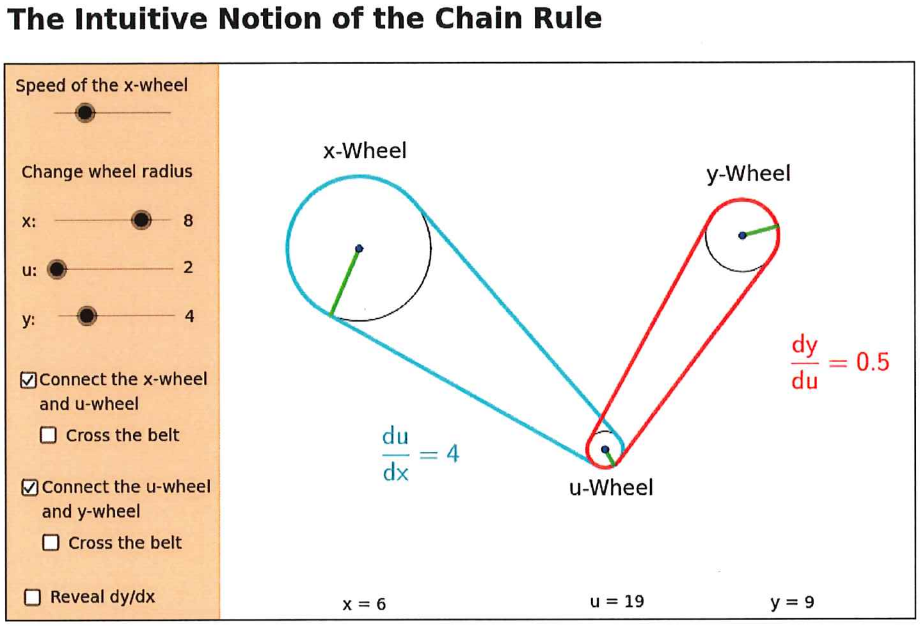
$$\frac{dM}{dS} = \frac{10\$}{\text{shirt}} \qquad \frac{dS}{dt} = 5\,\frac{\text{shirts}}{\text{purchase}}$$

- State the unknown: $\frac{dM}{dt}$
- Apply the chain rule:

$$\frac{dM}{dt} = \frac{dM}{dS}\frac{dS}{dt} = \frac{10\$}{\text{shirt}}\,\frac{5\text{ shirts}}{\text{purchase}} = \frac{50\$}{\text{purchase}}$$

Notes

An Online Demonstration



http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_intuitive_chain_rule.html

Notes

Using the Chain Rule

Question

Compute the derivative of $f(x) = \sqrt{2x+3}$.

- Identify the composition.

$$f(x) = \sqrt{2x+3} \Longrightarrow y = \sqrt{u} \qquad u = 2x+3$$

- Apply the chain rule:

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d\sqrt{u}}{du}\frac{d(2x+3)}{dx} = \frac{1}{2\sqrt{u}} \cdot 2 = \frac{2}{2\sqrt{2x+3}} = \frac{1}{\sqrt{2x+3}}$$

Notes

$$\frac{d}{dx}\left[y^2\right] = \text{"derivative of } y^2 \text{ with respect to } x\text{"}$$

The chain Rule

$$= \frac{d(y^2)}{dx}$$

$$= \left(\begin{matrix}\text{deriv of } y^2\\ \text{with respect to } y\end{matrix}\right)\left(\begin{matrix}\text{deriv of } y\\ \text{with respect to } x\end{matrix}\right)$$

$$= \frac{dy^2}{dy}\cdot\frac{dy}{dx} = (2y)\frac{dy}{dx}$$ Power Rule

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Composition

$$y^2 = (y(x))^2$$

$$\frac{dy^2}{dx} = \frac{d(y(x))^2}{d(y(x))}\frac{dy(x)}{dx}$$

$u = y(x)$

$$= \frac{du^2}{du}\cdot\frac{du}{dx}$$

$$= 2u\frac{du}{dx} = (2y)\frac{dy}{dx}$$

So far, all our functions were explicit.

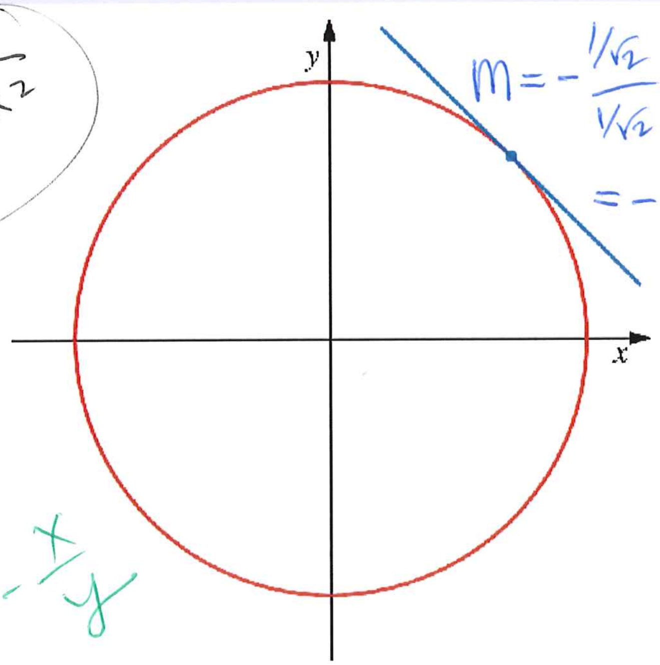
$f(x) = \dots$

Tangent to a Circle

Question

What's the tangent line to $x^2 + y^2 = 1$ when $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$?

$y = \pm\sqrt{1 - x^2}$



$m = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1$

An implicit relationship between variables can also be used to calculate $\frac{dy}{dx}$.

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$

https://www.desmos.com/calculator/as0jmnvbrt

The tangent line is $\left(y - \frac{1}{\sqrt{2}}\right) = (-1)\left(x - \frac{1}{\sqrt{2}}\right)$

Notes

Consider the relation $x^2 + y^2 = 1$.

$$\Rightarrow \frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[1] = 0$$

$$\Rightarrow \frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = 0$$

$$\Rightarrow 2x + \frac{dy^2}{dx} = 0$$

$$\Rightarrow 2x + \frac{dy^2}{dy}\frac{dy}{dx} = 0$$

Chain rule

THIS is the major use of the chain rule.

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y}$$

Tangent to a Circle

Question

What's the tangent line to $x^2 + y^2 = 1$ *when* $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$*?*

- Write y as a function of x.

$$y = \pm\sqrt{1 - x^2}$$

- Determine that you want $y > 0$ because $y = \frac{1}{\sqrt{2}}$. This gives $y = \sqrt{1 - x^2}$.
- Calculate the derivative using the chain rule:

$$\frac{dy}{dx} = \frac{d\sqrt{1-x^2}}{d(1-x^2)} \frac{d(1-x^2)}{dx} = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

- Evaluate at $x = \dfrac{1}{\sqrt{2}}$.

$$\frac{dy}{dx} = \overbrace{\frac{-1/\sqrt{2}}{\sqrt{1-(1/\sqrt{2})^2}}}^{\text{Eeep!}} = \frac{-1/\sqrt{2}}{\sqrt{1/2}} = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1$$

Notes

Tangent to a Circle

Question

What's the tangent line to $x^2 + y^2 = 1$ when $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ assuming $m_{tan} = -1$?

- We know $m_{tan} = -1$ and $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is on the line.
- Apply point-slope format of lines:

$$y - y_0 = m(x - x_0) \iff y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right)$$

- Do some algebra to make things look nicer:

$$y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right) \iff x + y = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Notes

Tangent to a Circle

Question (Again! This time, with feeling!)

What's the tangent line to $x^2 + y^2 = 1$ *when* $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$*?*

- Write y as a function of x. Call it $y = y(x)$.
- Differentiate both sides of $x^2 + y^2 = 1$.

$$\frac{d}{dx}\left[x^2 + (y(x))^2\right] = \frac{d}{dx}[1] \iff 2x + \frac{d}{dx}\left[(y(x))^2\right] = 0$$

- Apply the chain rule to $(y(x))^2$.

$$\frac{d}{dx}\left[(y(x))^2\right] = \frac{d(y(x))^2}{dy(x)}\frac{dy(x)}{dx} = 2y(x)y'(x)$$

- We now have $2x + 2y(x)y'(x) = 0$. Solve for $y'(x)$.

$$y'(x) = \frac{-2x}{2y(x)} = \frac{-x}{y} \iff \frac{dy}{dx} = \frac{-x}{y} = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1$$

Notes

The Magic of the Chain Rule

Everywhere y appears in the formula, differentiation will produce $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[x^2 + y^2\right] = 2x + 2y\frac{dy}{dx}$$

Amazing!

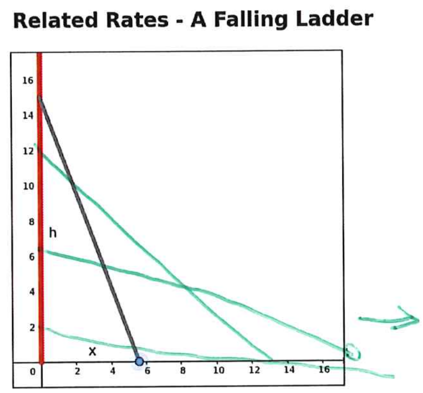
$$\frac{dy^2}{dx} = \frac{dy^2}{dy}\frac{dy}{dx} = (2y)\frac{dy}{dx}$$

Notes

The Sliding Ladder

Question

A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?



✓ Introduce notation

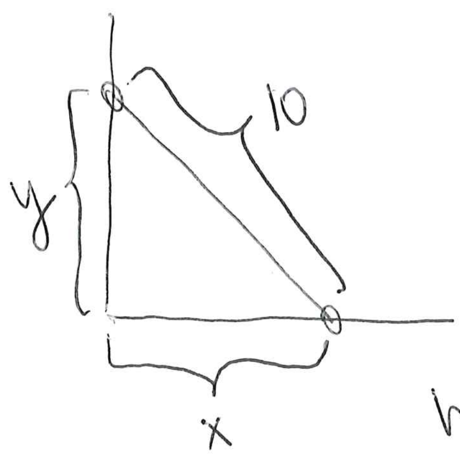
✓ Relate variables

○ take derivatives.

http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_app_rr_falling_ladder.html

Notes

~~We know~~ Let x be the distance in feet ~~meters~~ from base of ladder to wall. Let y be distance from top of ladder to ground in feet.



We know, by Pythagoras,

$$x^2 + y^2 = 10^2 = 100$$

We have: $\frac{dy}{dt} = -2$, $x = 5$

We want: $\frac{dx}{dt} = ?$

We calculate:

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[10^2\right] = 0$$

The Sliding Ladder

Question

A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?

- Introduce variables:
  $y =$ top of ladder in ft, $x =$ bottom of ladder in ft, $\ell =$length of ladder in ft
  $t =$ time in seconds
- State the known information:

$$\frac{dy}{dt} = -2ft/s \qquad x = 5ft \qquad \ell = 10ft$$

- State the unknown: $\frac{dx}{dt}$
- Relate the variables:

$$x^2 + y^2 = \ell^2 \iff x^2 + y^2 = 100$$

Notes

The Sliding Ladder

Question (Fully mathematized!)

Find dx/dt assuming:

$$\frac{dy}{dt} = -2ft/s \qquad x = 5ft \qquad \ell = 10ft \qquad x^2 + y^2 = 100$$

- Differentiate both sides.

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}[100] \longleftrightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

- Apply the given information.

$$\frac{dx}{dt} = \frac{4y}{10} = \frac{2y}{5} = 2\sqrt{3}$$

- Solve for the new unknown y.

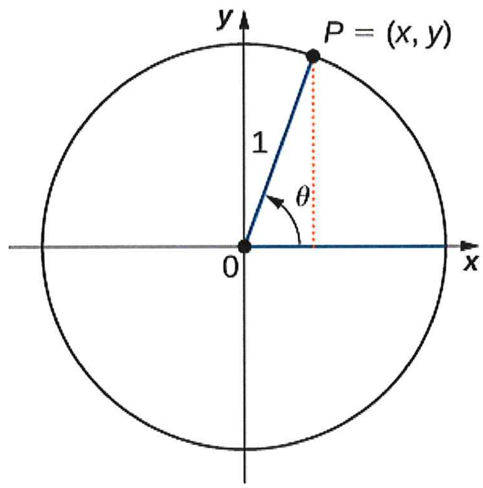
$$x^2 + y^2 = 100 \longleftrightarrow 5^2 + y^2 = 100 \longleftrightarrow y = \pm\sqrt{75} = \pm 5\sqrt{3}$$

Notes

The Trigonometric Functions sin and cos

Question

How are $\sin(x)$ *and* $\cos(x)$ *related?*

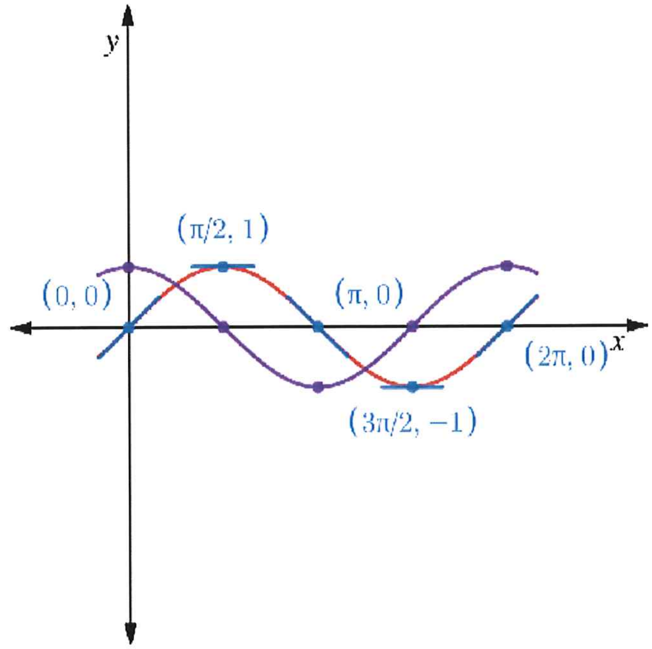


OpenStax §1.3 Fig 1.31

$$(x, y) = (\cos(\theta), \sin(\theta)) \iff \sin^2(\theta) + \cos^2(\theta) = 1$$

Notes

Tangents to the sin Curve



$y = \sin(x)$ with derivative curve

Notes

Tangents to the sin Curve

Theorem

$$\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$$

Notes

What's the Derivative?

Question

Which of the following is the derivative of $y = x^2 \sin(x)$*?*

1. $\frac{dy}{dx} = 2x\sin(x) + x^2\cos(x)$
2. $\frac{dy}{dx} = 2x\cos(x)$
3. $\frac{dy}{dx} = x^2\cos(x)$
4. $\frac{dy}{dx} = 2x\cos(x) + x^2\sin(x)$

Notes

The Trigonometric Functions sin and cos

Question

What does $\sin^2(x) + \cos^2(x) = 1$ *say about derivatives?*

$$\sin^2(x) + \cos^2(x) = 1 \qquad \text{\# write out the relationship}$$

$$\frac{d}{dx}\left[\sin^2(x) + \cos^2(x)\right] = \frac{d}{dx}[1] \qquad \text{\# take derivatives of both sides}$$

To take the derivative of both sides,we must calculate:

$$\frac{d}{dx}\left[\sin^2(x)\right] = \frac{d}{dx}[\sin(x)]\sin(x) + \sin(x)\frac{d}{dx}[\sin(x)] = 2\frac{d}{dx}[\sin(x)]$$

The same sort of calculation for cos gives:

$$\frac{d}{dx}\left[\cos^2(x)\right] = 2\frac{d}{dx}[\cos(x)]$$

Notes

The Trigonometric Functions sin and cos

When we take derivatives of both sides of $\sin^2(x) + \cos^2(x) = 1$ we get:

$$2\frac{d}{dx}[\sin(x)]\sin(x) + 2\frac{d}{dx}[\cos(x)]\cos(x) = 0$$

This looks very fancy, but it is saying: $2ss' + 2cc' = 0 \iff ss' = -cc'$.

Our graphical investigation showed up: $\frac{d}{dx}[\sin(x)] = \cos(x)$.

So, we can re-write this as:

$$\sin(x)\underbrace{\frac{d}{dx}[\sin(x)]}_{\text{known}} = -\cos(x)\underbrace{\frac{d}{dx}[\cos(x)]}_{\text{want to know}}$$

This allows us to conclude:

$$\sin(x)\cos(x) = -\cos(x)\frac{d}{dx}[\cos(x)]$$

Thus, $\frac{d}{dx}[\cos(x)] = -\sin(x)$.

Notes

Implicit Differentiation

Question

Find $\frac{dy}{dx}$ *if* $3x^3 + 9xy^2 = 5x^3$.

- Differentiate both sides.

$$\frac{d}{dx}\left[3x^3 + 9xy^2\right] = \frac{d}{dx}\left[5x^3\right] \iff 9x^2 + 9y^2 + 18xy\frac{dy}{dx} = 15x^2$$

- Isolate for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{15x^2 - 9x^2 - 9y^2}{18xy} = \frac{6x^2 - 9y^2}{18xy} = \frac{2x^2 - 3y^2}{6xy}$$

Notes

Related Rates or Applied Chain Rule

Words → Math

- Introduce variables.
- State the known information
- State the unknown.
- Relate the variables.

Math → Victory!

- Differentiate both sides.
- Apply the given information.
- Solve for the new unknowns.
- Solve the problem!

Notes

The Lighthouse

Question (OS §4.1 Q10)

A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach. If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?

- Introduce variables.
  $x =$ distance from light on shore to P in miles, $t =$ time in minutes
  $\theta =$ angle of light from LP measured clockwise in radians
- State the known information

$$x = 2 \qquad \frac{d\theta}{dt} = \frac{10 \text{ revolutions}}{\text{minute}} = 10 \times 2\pi = 20\pi \text{ radians per minute}$$

- State the unknown.

$$\frac{dx}{dt} = \text{rate that the beam moves across the beach in miles per minute}$$

Notes

The Lighthouse

Question (OS §4.1 Q10)

A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach. If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min $= 20\pi$ *, how fast does the beam of light move across the beach* $x = 2$ *mi away from the closest point on the beach?*

- Relate the variables.

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{x}{4} \iff \frac{\sin(\theta)}{\cos(\theta)} = \frac{x}{4} \iff 4\sin(\theta) = x\cos(\theta)$$

- Differentiate both sides.

$$\frac{d}{dt}\left[4\sin(\theta)\right] = \frac{d}{dt}\left[x\cos(\theta)\right] \iff 4\cos(\theta)\frac{d\theta}{dt} = \frac{dx}{dt}\cos(\theta) - x\sin(\theta)\frac{d\theta}{dt}$$

- Apply the given information.

$$\iff 4\cos(\theta)20\pi = \frac{dx}{dt}\cos(\theta) - 2\sin(\theta)20\pi$$

Notes

The Lighthouse

Question (OS §4.1 Q39)

A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach. If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach $x = 2$ mi away from the closest point on the beach?

- Solve for the new unknowns.
  We need some geometry! If $PX = 2$ and $LP = 4$ then Pythagoras gives us: $LX = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. The trigonometric ratios give us:

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}} \qquad \sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$$

  Putting this together gives:

$$4\cos(\theta)20\pi = \frac{dx}{dt}\cos(\theta) - 2\sin(\theta)20\pi \iff 4\frac{2}{\sqrt{5}}20\pi = \frac{dx}{dt}\frac{2}{\sqrt{5}} - 2\frac{1}{\sqrt{5}}20\pi$$

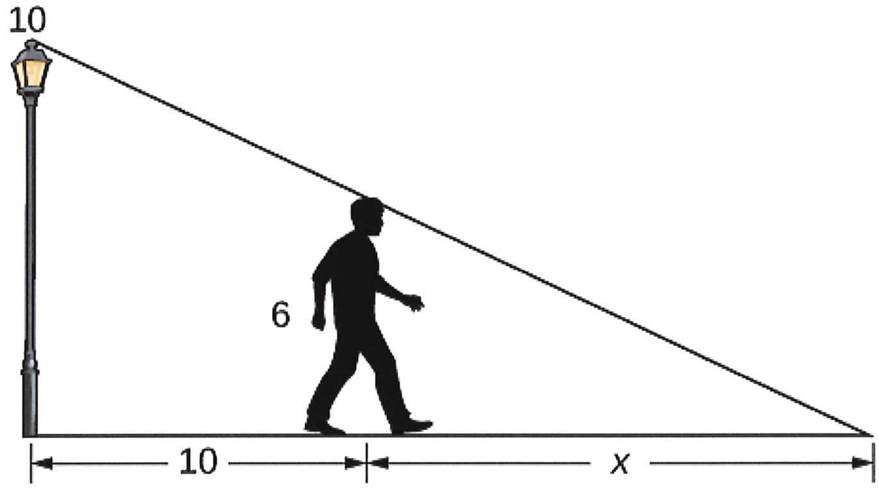
- Solve the problem! $\dfrac{dx}{dt} = 100\pi\text{mi} \,/\, \text{min}$

Notes

The Growing Shadow

Question (OS §4.1 Q10)

A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?



Notes

The Growing Shadow

Question (OS §4.1 Q10)

A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?

- Introduce variables.
  $\ell =$ distance from pole in ft, $x =$ length of shadow in ft
  $h =$ height of person, $t =$ time in seconds.
- State the known information: $\ell = 10 \quad h = 6 \quad \dfrac{d\ell}{dt} = 3$
- State the unknown: $\dfrac{dx}{dt} =$ the rate of change of the length of the shadow.
- Relate the variables. By similar triangles, we have:

$$\frac{x}{6} = \frac{x + \ell}{10}$$

Notes

The Growing Shadow

Question (OS §4.1 Q10 → Math!)

Find $\frac{dx}{dt}$ *assuming:*

$$\frac{x}{6} = \frac{x+\ell}{10} \quad \ell = 10 \quad h = 6 \quad \frac{d\ell}{dt} = 3$$

- Differentiate both sides.

$$\frac{d}{dt}\left[\frac{x}{6}\right] = \frac{d}{dt}\left[\frac{x+\ell}{10}\right] \iff \frac{1}{6}\frac{dx}{dt} = \frac{1}{10}\frac{dx}{dt} + \frac{1}{10}\frac{d\ell}{dt}$$

- Apply the given information.

$$\left(\frac{1}{6} - \frac{1}{10}\right)\frac{dx}{dt} = \frac{1}{10}\frac{d\ell}{dt} = \frac{1}{10}3$$

- Solve the problem!

$$\frac{dx}{dt} = \frac{3/10}{1/6 - 1/10} = \frac{9}{2}$$

Notes

Summary of Week 5

- tk: add summary

Notes