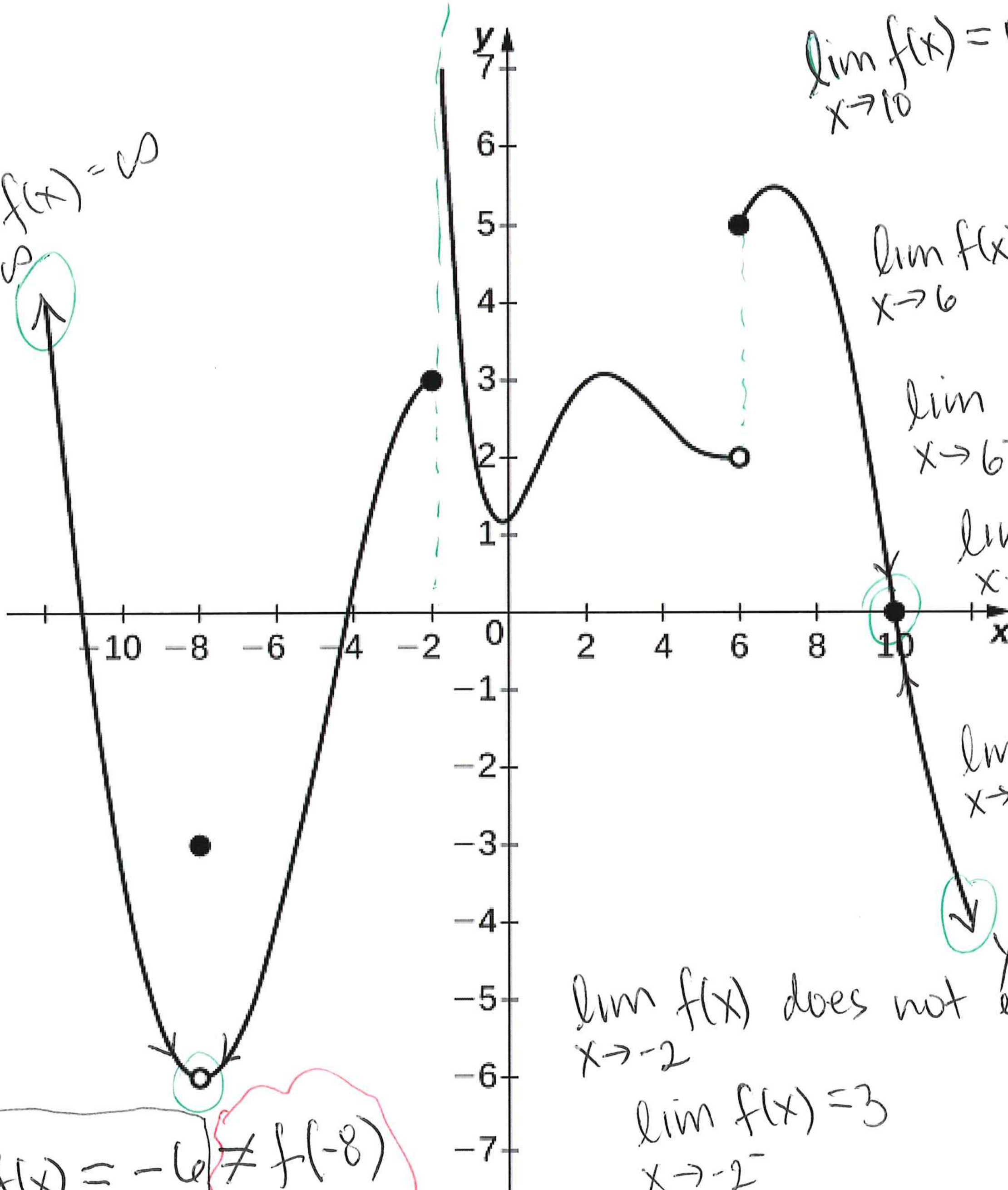
Activity

Question: A Bunch of Limits

Analyze the following graph using the notion of limits.



$\lim_{x \to -\infty} f(x) = \infty$

$\lim_{x \to 10} f(x) = 0$

$\lim_{x \to 6} f(x)$ does not exist!

$\lim_{x \to 6^+} f(x) = 5$

$\lim_{x \to 6^-} f(x) = 2$

$\lim_{x \to \infty} f(x) = -\infty$

$\lim_{x \to -2} f(x)$ does not exist!

$\lim_{x \to -2^-} f(x) = 3$

$\lim_{x \to -2^+} f(x) = \infty$

$\boxed{\lim_{x \to -8} f(x) = -6} \neq f(-8) = -3$

This limit exists

f not continuous at $x = -8$

Micro-Assignment

Activity: Check-In: 5 min

Please answer the following on a sheet of paper.

1. How is first year going?
2. How do you feel about this course?
3. What is something that you want to know?

Parker will walk around the room and chat with people.
Hand in your paper when you're done.

Summary of Week 2

- Polynomials: leading coefficients, degree
- Even and odd symmetry
- Composition
- Limits

Week 2: Polynomials

Definition: Polynomials — OS §1.2 Eq. 1.7

A **polynomial** is a function of the form:

leading coeff.

degree

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

The **degree** d of a polynomial is the highest power in it. $= n$
The number in front of the highest power is the **leading coefficient**. $= a_n$

Example: Coefficients

Write out the coefficients of:

$a_3 = 2 \quad a_2 = 9 \quad a_1 = 0 \quad a_0 = 0.$

$$\boxed{f(x) = 2x^3 + 9x^2} = 2x^3 + 9x^2 + 0x + 0.$$

Definition: End Behaviour of Polynomials

The end behaviour of $f(x)$ is what $f(x)$ does when x gets very large ($x \to \infty$). or ($x \to -\infty$)

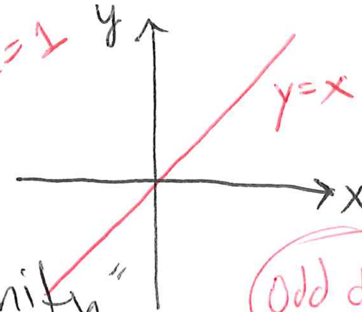
The end behaviour of $f(x) = x^1$:

$f(x) \to \infty$ as $x \to \infty$

and $f(x) \to -\infty$ as $x \to -\infty$.

"f(x) goes to infinity as x goes to infinity"

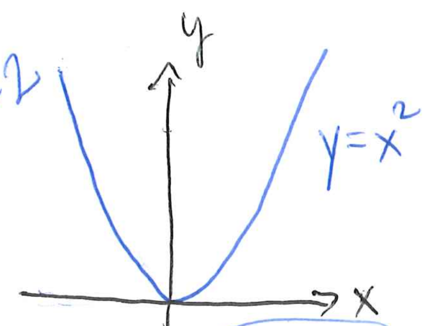
degree = 1



Odd deg

The ... $= x^2$

degree = 2



Even deg

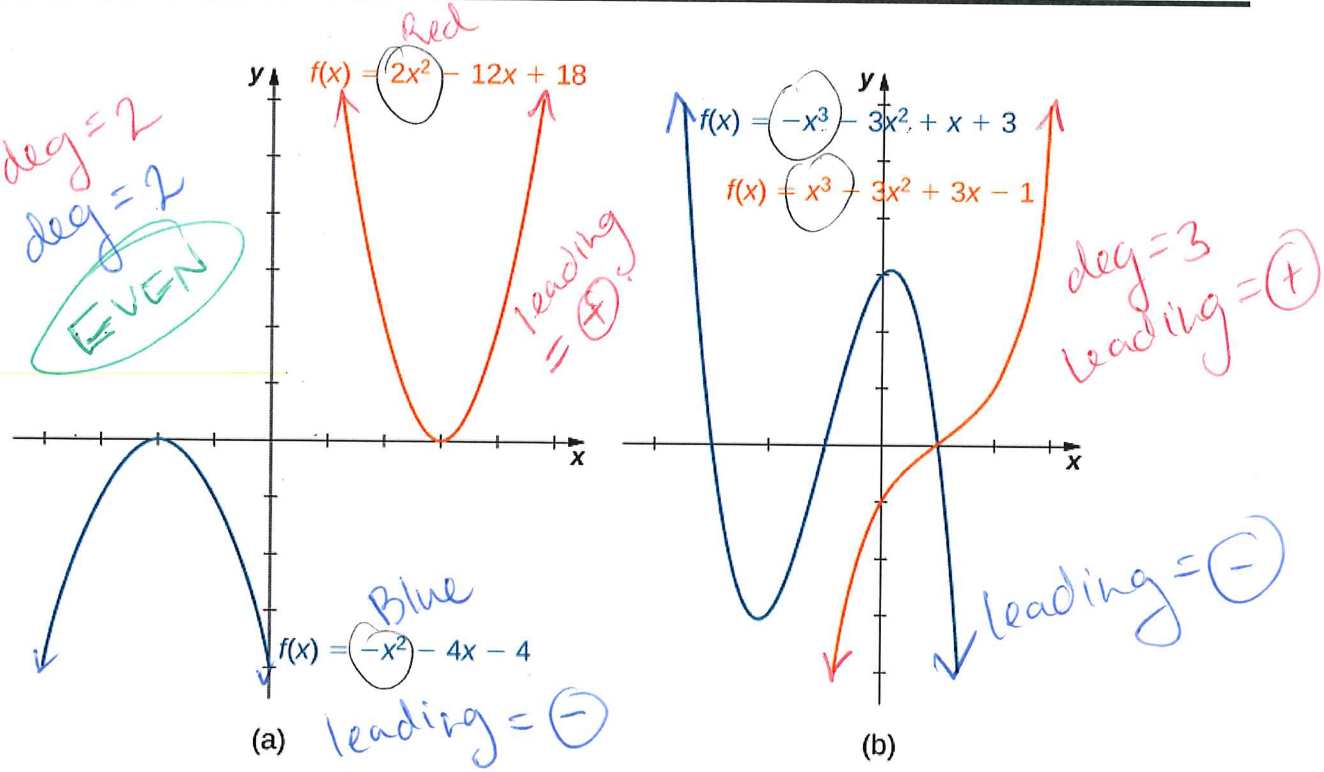
Week 2

Back of page 25 ~~missing~~. Not on Quercus.

)

For ... 1 000 000.

$= (-1000)^2$



OpenStax §1.2 Figure 1.19

Example: End Behaviour Visually

Describe the end behaviour of the functions above.

AND

(a) $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.

$f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

These show we can't figure out end behaviour using JUST degree. We need to consider leading coeff.

(b) $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

$f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.

The end behaviour of a polynomial is COMPLETELY determined by

even/odd of degree and +/− of leading coeff.

Question: Describe End Behaviour

Using the words "DEGREE d" and "LEADING COEFFICIENT a" make a short paragraph which describes the end behaviour of polynomials of even degree.

If a polynomial has even degree, then DEGREE d is even.

If the LEADING COEFFICIENT $a > 0$ then $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.

If the LEADING COEFFICIENT $a < 0$ then $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

Activity: Micro-Assignment: 5 min

Use the words

- DEGREE d
- LEADING COEFFICIENT a
- ~~THE~~ $x \to \infty$ ~~END~~
- ~~THE~~ $x \to -\infty$ ~~END~~

to make a short paragraph explaining the end behaviour of polynomials of odd degree.

$-f(x) = f(-x)$

Definition: Symmetry

A graph has **even symmetry** if $f(x) = f(-x)$. It has **odd symmetry** if $f(x) = -f(-x)$.

Example: Even and Odd

The functions $f(x) = x^2,\ x^4,\ x^6$ have even symmetry. The functions $f(x) = x,\ x^3,\ x^5$ have odd symmetry.

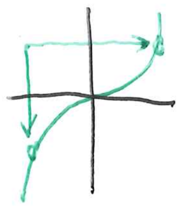
$f(x) = x^2 \Rightarrow f(x) = f(-x) = (-x)^2 = x^2$

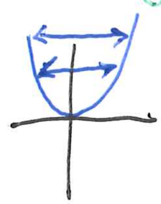
$f(x) = x^3 \Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x)$

Activity: Discuss: 1 min

Why are these called even and odd symmetries?

- odd symmetry around origin
- even symmetry about y-axis.





- even and odd of degree determines symm.

(!) even degree does not imply even symm.

We need: all terms in polynomial have even exponents. $\Rightarrow$ even symm

We need: all terms in poly. have odd exponents. $\Rightarrow$ odd symm

$\rightarrow f(x) = x^2 + 1 \cdot x^0$

Ex: Is $\boxed{f(x) = x^2 + 1}$ even or odd?

$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$

Therefore, $f(x)$ is even.

<u>Notice</u>: The exponents 2 and 0 are both even.

		Even: Yes	Even: No
Odd	Yes	0	x^3
	No.	x^2	$x^3 + x^2$ $x + 1$ $x^2 + x$

Activity: Make Some Examples: 5 min

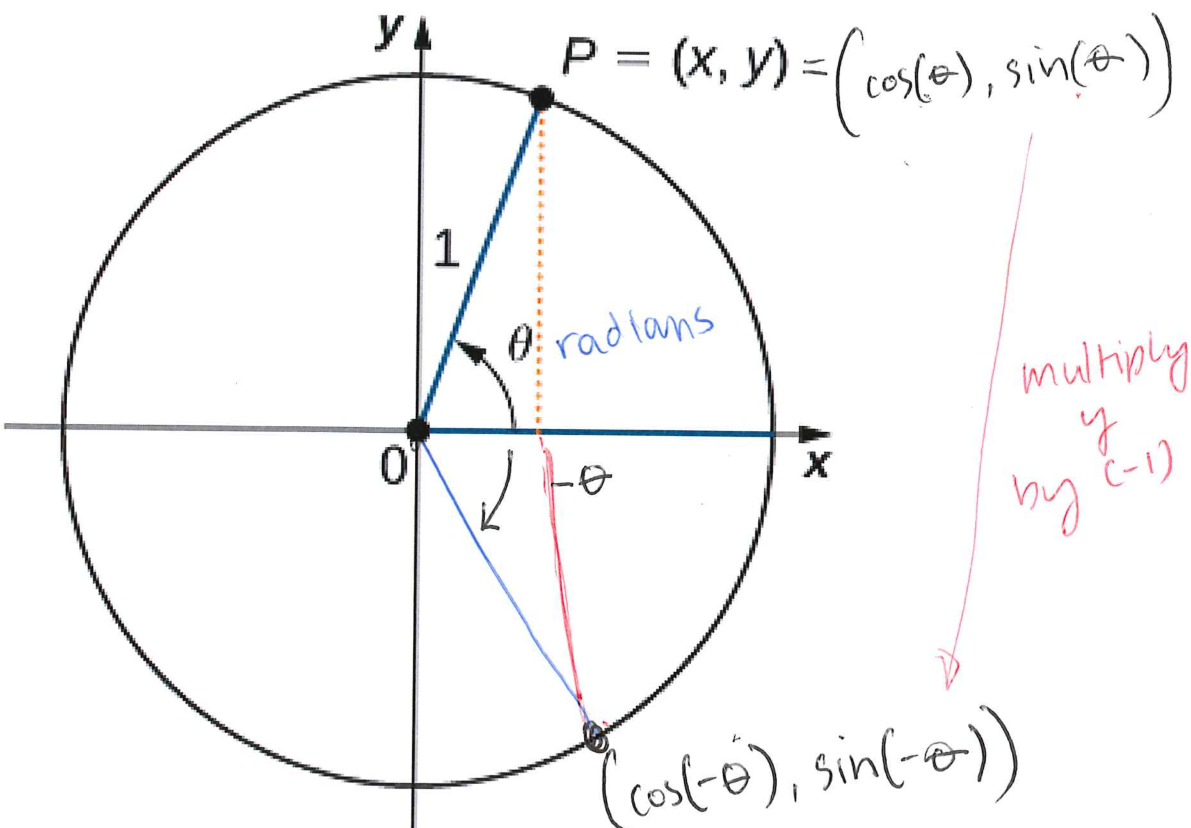
Whenever we meet two similar mathematical properties, it is important to investigate how they are related by finding examples. Find an example of $f(x)$ for each box.

		Even: Yes	Even: No
Odd	Yes		
	No		

Definition: The Trigonometric Functions

On the unit circle $x^2 + y^2 = 1$ we have:

$$(x, y) = (\cos(\theta), \sin(\theta)) = (\cos(-\theta), -\sin(-\theta))$$



OpenStax §1.3 Fig 1.31

This gives: $\cos(\theta) = \cos(-\theta)$ ad
$\sin(\theta) = -\sin(-\theta)$.

Therefore: $\cos(\theta)$ is even ad
$\sin(\theta)$ is odd.

Question: Working Backwards

Write the function $f(x) = \dfrac{1}{x^2+3}$ as a composition $(a \circ b \circ c)(x)$ for three functions: $a(x), b(x), c(x)$.

$$\boxed{f(x) = \frac{1}{x^2+3}} = \frac{1}{(((x)^2)+3)} = \frac{1}{(((x)^2)+3)}$$

$c(x) = x^2$

$b(x) = x+3$

$a(x) = \dfrac{1}{x}$

For this composition:

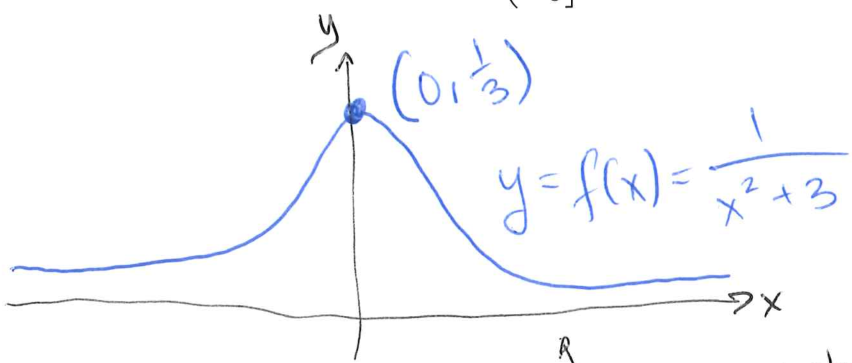
$$\begin{aligned}(a \circ b \circ c)(x) &= a(b(c(x))) \\ &= a(b(x^2)) \\ &= a(x^2+3) \\ &= \frac{1}{x^2+3}\end{aligned}$$

Yay! It matches.

Question: A Complicated Range

Find the range of $f(x) = \dfrac{1}{x^2+3}$.

Graph it in Desmos. Notice that the graph is contained in $R = \left(0, \dfrac{1}{3}\right]$. How do we explain this?



The graph never touches x-axis.

The range of $c(x)$ is $[0, \infty)$. # $c(x) = x^2$

The range of $b(c(x))$ is $[3, \infty)$ # $b(c(x)) = x^2+3$

The range of $a(b(c(x)))$ is "$\left(\frac{1}{\infty}, \frac{1}{3}\right]$" # $a(b(c)) ~~is~~ \frac{1}{x^2+3}$.

(!) This notation $\frac{1}{\infty}$ is BAD. Don't treat infinity as a number.

The range of $a(b(c(x)))$ is $\left(0, \frac{1}{3}\right]$ because $a(b(c(0))) = \frac{1}{3}$ and the end behaviour of $a(b(c(x)))$ is $\longrightarrow 0$.

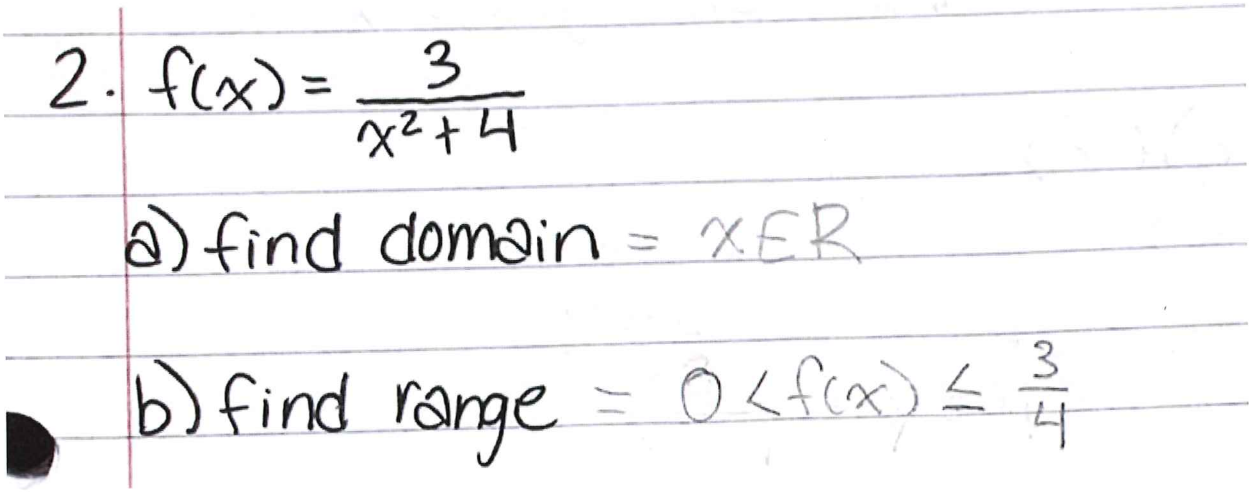
Why is "$\frac{1}{\infty} = 0$"? For any large finite number N we have $\frac{1}{N}$ is small.

Remark: An Important Observation

Look at the following pair of solutions to the problem.

$$\text{Find the domain of } f(x) = \frac{3}{x^2+4}.$$

What do you notice about the ratio of words to symbols?
Who do you think would score a higher mark?

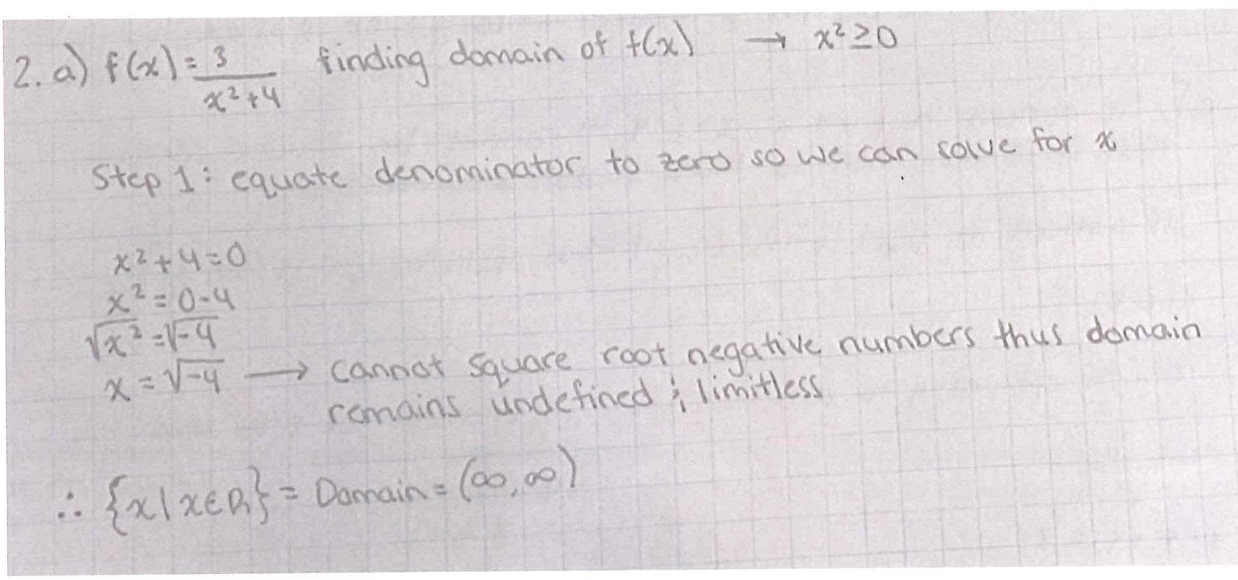


2. $f(x) = \dfrac{3}{x^2+4}$

a) find domain $= x \in R$

b) find range $= 0 < f(x) \leq \dfrac{3}{4}$

Student Solution #1



2. a) $f(x) = \dfrac{3}{x^2+4}$ finding domain of $f(x)$ $\rightarrow$ $x^2 \geq 0$

Step 1: equate denominator to zero so we can solve for x

$x^2+4=0$
$x^2=0-4$
$\sqrt{x^2}=\sqrt{-4}$
$x=\sqrt{-4}$ $\longrightarrow$ cannot square root negative numbers thus domain remains undefined & limitless

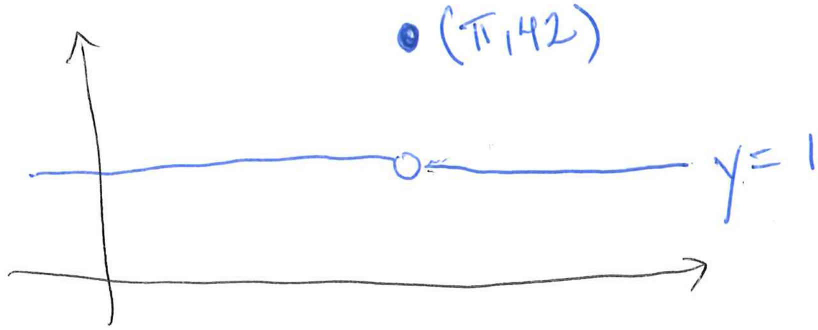
$\therefore \{x \mid x \in R\} = \text{Domain} = (\infty, \infty)$

Student Solution #2

Question: A Piecewise Range

Write the domain and range of the function.

$$f(x) = \begin{cases} 1 & x \neq \pi \\ 42 & x = \pi \end{cases}$$



Domain: The pieces $x \neq \pi$ ad $x = \pi$ cover all possible real numbers. Therefore, the domain is $\mathbb{R}$.

Range: The function only outputs $y=1$ and $y=42$. Therefore, the range is $\{1, 42\}$.

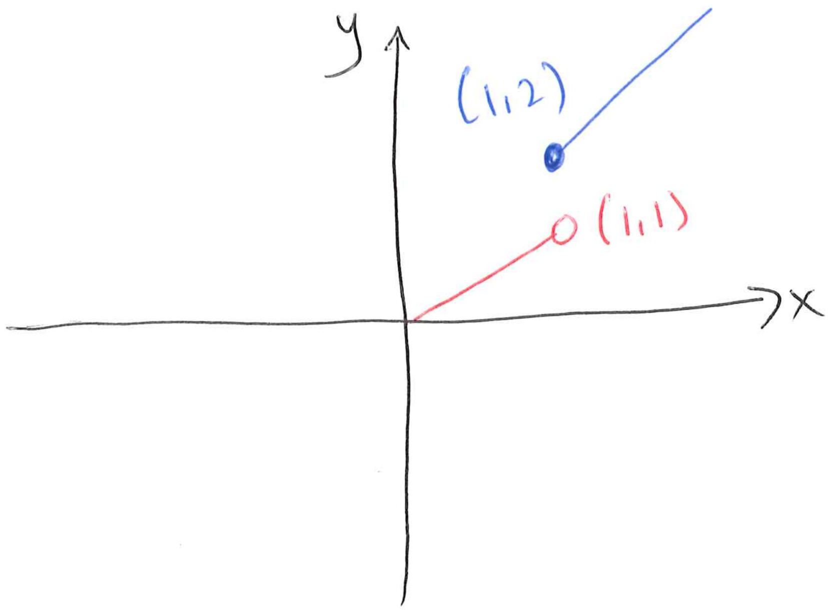
The curly brackets $\{ \ldots \}$ denote sets. The set $\{1, 42\}$ has exactly two elements $y=1$ ad $y=42$.

Activity: Solo Work: 2 min

Sketch the function:

$$f(x) = \begin{cases} x & x < 1 \\ x+1 & x \geq 1 \end{cases}$$

What do you notice about the values of $f(x)$ near $x = 1$?

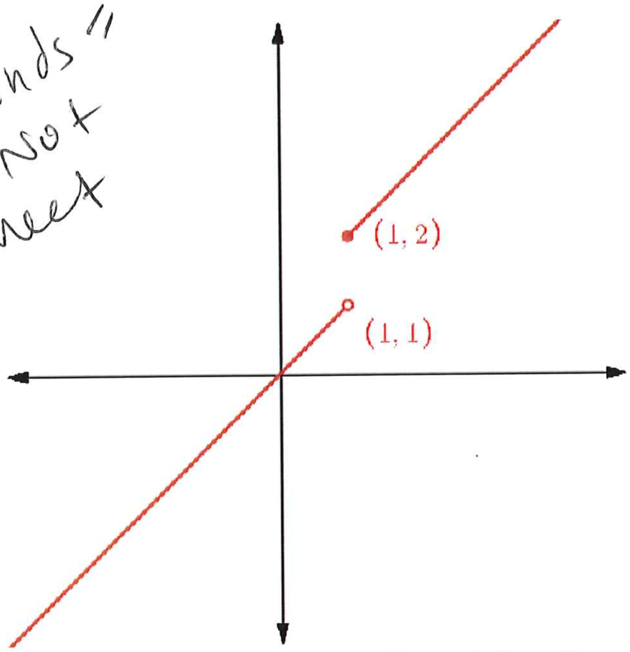


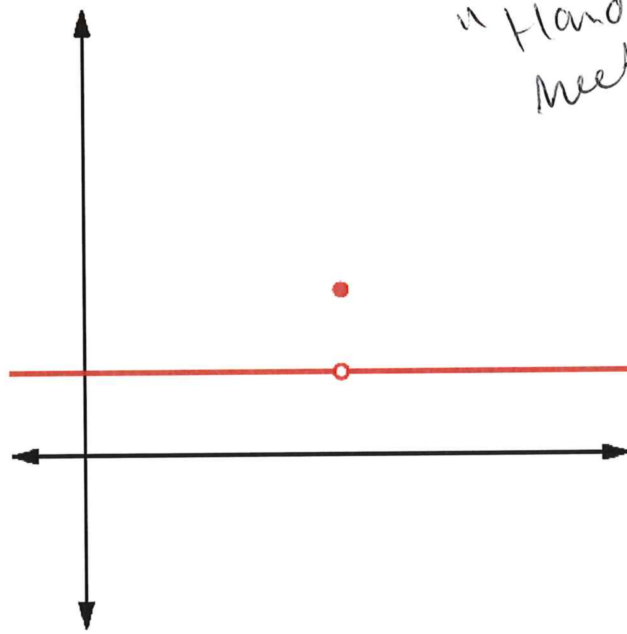
Notice: The lines do not meet!
There is a jump/gap.

Remark: Three Similar Problems

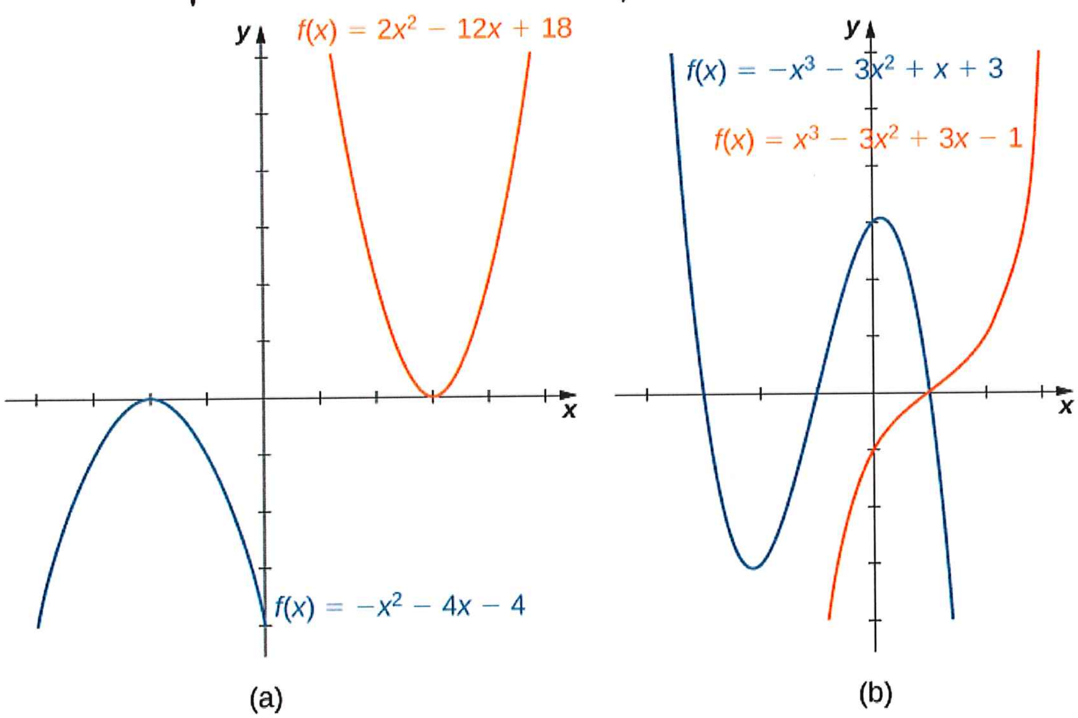
One of the "super powers" of mathematics is its ability to explain multiple phenomena using the same language. This is unifying process is called abstraction or generalization.

"Hands" Do Not Meet





"Hands" Meet



End Behaviour

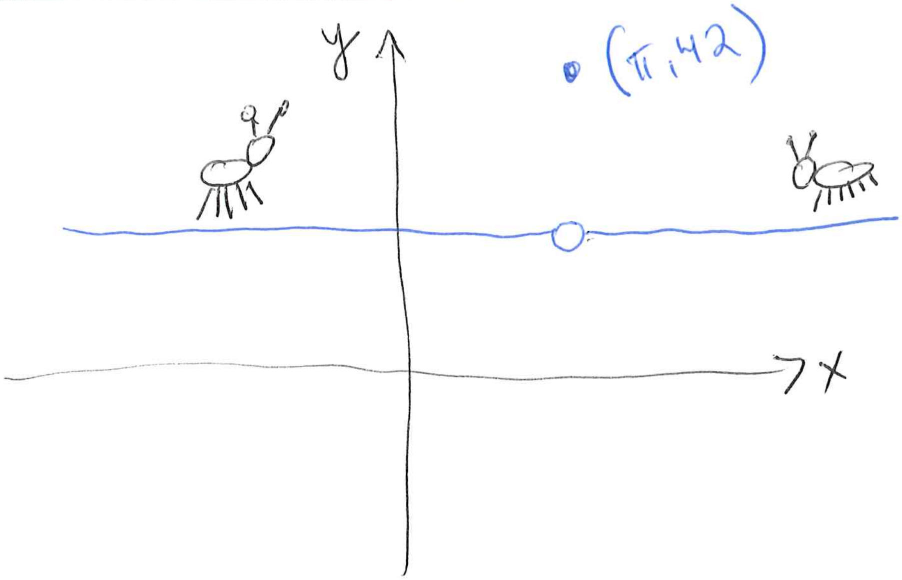
Definition: Limits **OS §1.2 Defn 2.3**

If $f(x)$ gets closer to L as x gets closer to a then $\lim_{x\to a} f(x) = L$.
We say that "the limit of $f(x)$ as x approaches a is L".

Question: A Strange Limit

What is the limit of $f(x)$ as x approaches π?

$$f(x) = \begin{cases} 1 & x \neq \pi \\ 42 & x = \pi \end{cases}$$



As we approach $x=\pi$, the graph is always at height $y=1$.

We get: $\lim_{x\to\pi} f(x) = 1$

(The point $(\pi, 42)$ does not effect limit.)

Question: Limits Visually

Compute the limit $\lim_{x \to 0} \frac{\sin(x)}{x}$ by inspecting the graph on Desmos.

(!) We can't always find limits by plugging in.

$$" \frac{\sin(0)}{0} " \to " \frac{0}{0} " \leftarrow \text{undefined.}$$

Inspecting the graph on Desmos, we see $\frac{\sin(x)}{x}$ is undefined at $x=0$.

When we zoom in near $x=0$, we see the graph looks like $y=1$.

Therefore,

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

What happens if $f(x)$ is defined at $x=0$?

We may have $\lim_{x \to 0} f(x) = f(0)$ ← f is continous at $x=0$

but sometimes $\lim_{x \to 0} f(x) \neq f(0)$

Question: Limits Algebraically

Compute the limit $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

First note: we cannot plug in $x=1$ because

$$\text{"}\frac{1^2-1}{1-1}\text{"} = \text{"}\frac{0}{0}\text{"} \leftarrow \text{undefined.}$$

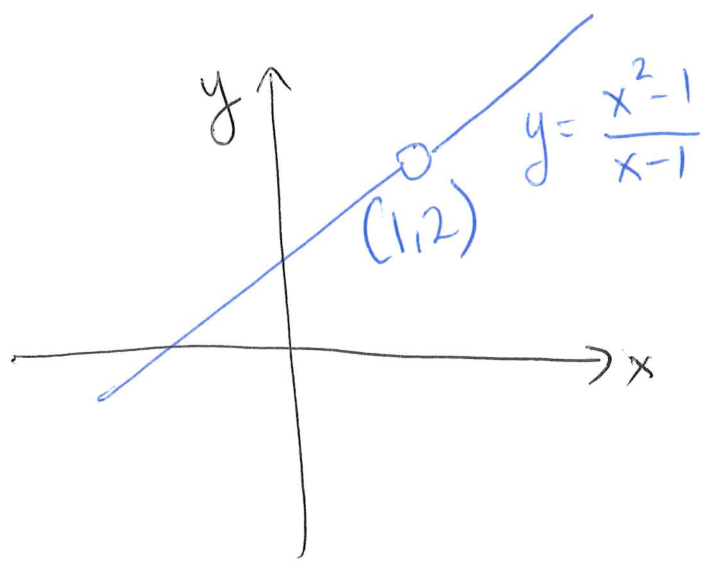
we can factor:

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)}$$

For all $x \neq 1$ we can cancel.

$$= \lim_{x \to 1} x+1$$

$$= 1+1 = 2$$



t^- t t^+

Definition: Two Sided Limits

Every number t has two sides:

the **left side** of t which we write t^- and the **right side** of t which we write t^+.

The left hand limit is $\lim_{x \to t^-} f(x) = L$ if $f(x)$ approaches L when $x < t$.
The **right hand limit** is $\lim_{x \to t^+} f(x) = L$ if $f(x)$ approaches L when $x > t$.

Question: A Jump

The function $f(x)$ has a "jump" from $(1, 1)$ to $(1, 2)$.

$$f(x) = \begin{cases} x & x < 1 \\ x + 1 & x \geq 1 \end{cases}$$

Describe the nature of this jump using limits.

We have:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x \quad \text{\# } x<1 \text{ so we use } f(x) = x$$

$$= 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x+1 \quad \text{\# } x>1 \text{ so we use } f(x) = x+1$$

$$= 1+1 = 2$$

There is a jump from (1,1) to (1,2) because the left hand and right hand limits give those values.

Theorem: The Limit Laws

1. $\lim_{x \to a} c = c$
2. If n is a non-negative integer, then $\lim_{x \to a} x^n = a^n$.
3. $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
4. $\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
5. $\lim_{x \to a} f(x) \times g(x) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$
6. If $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x)/$
7. If $f(x) = g(x)$ for $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g($

Note: You will be provided these laws on the term test

Week 3 + week 2 - end
↓
scan back
of pg. 38

Example: Using the limit laws

Calculate and justify the following limit using the limit

$$\lim_{x \to 1} \frac{x^2 - 3x + 7}{x + 1} = \frac{1 - 3 \cdot 1 + 7}{1+1} = \frac{5}{2}$$

Note: This is long and boring, but we need to do it.

We apply 6: We check $\lim_{x \to 1} x+1 \neq 0$.

By 3 we have:

$$\lim_{x \to 1} x+1 = \lim_{x \to 1} x + \lim_{x \to 1} 1 = 1 + 1 = 2$$

By 2 we have:

$$\lim_{x \to 1} x = 1$$

By 1 we have:

$$\lim_{x \to 1} 1 = 1$$

Therefore, $\lim_{x \to 1} x+1 = 2 \neq 0$. (Back of p.38)

Definition: Does Not Exist

If $f(x)$ does not get close to any number L as x approaches a then the limit $\lim_{x\to a} f(x)$ does not exist (DNE). Alternatively, the limit is not well defined.

- *Why do we say "does not exist"?*
  We say this because there is no value L that the function gets close to.
  The value L is the thing which does not exist.

- *Why do we say "the limit is not well-defined"?*
  We say this because the notation $\lim_{x\to a} f(x)$ is ambiguous if:

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$$

- These two terms are equivalent.
  You probable heard "DNE" in highschool
  Parker might say "is not well-defined".

we get: $\lim_{x\to 1} \frac{x^2-3x+7}{x+1} = \frac{\left(\lim_{x\to 1} x^2-3x+7\right)}{\left(\lim_{x\to 1} x+1\right)}$

$= \frac{\left(\lim_{x\to 1} x^2-3x+7\right)}{2} = \frac{5}{2}$

By ③ ad ④ we get:

$\lim_{x\to 1} x^2-3x+7 = \left(\lim_{x\to 1} x^2\right) - \left(\lim_{x\to 1} 3x\right) + \left(\lim_{x\to 1} 7\right)$

$= \left(\lim_{x\to 1} x^2\right) - \left(\lim_{x\to 1} 3x\right) + 7$ # ① constants

$= 1^2 - \left(\lim_{x\to 1} 3x\right) + 7$ # ② polynomials

$= 8 - \left(\lim_{x\to 1} 3\right)\left(\lim_{x\to 1} x\right)$ # ⑤ multiplication.

$= 8 - 3\cdot 1$ # ① constants ad ② polynomial.

$= 5$