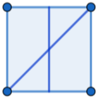
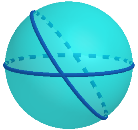
MAT 402: Classical Geometry

Groups

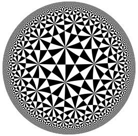


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

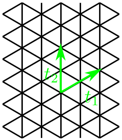
Spherical



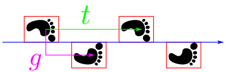
Hyperbolic



Tilings



Friezes

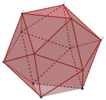


Trigonometry

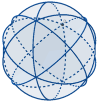
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

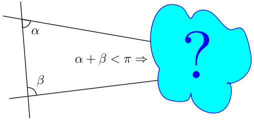
Platonic Solids

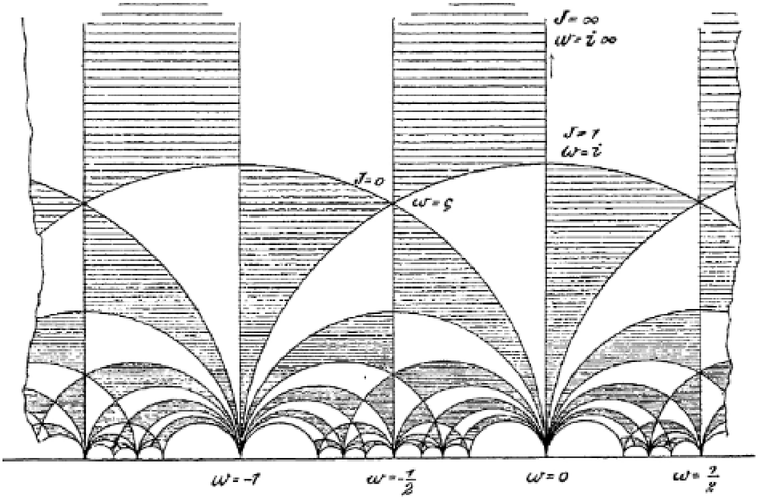


Coxeter



Parallels





From Klein and Fricke - Theorie der Elliptischen Modulfunction (1927)

Homework #5 and Model #5 Extended. Feedback #4 is out.

MAT 402: Friday November 20th 2020

Learning Objectives:

- Use complex coordinated to describe the upper-half plane model.
- Find fractional-linear transformations with specific properties.

Fractional Linear Transformations

Theorem

A fractional linear map of the form:

$$\frac{az+b}{cz+d} \quad or \quad \frac{a\bar{z}+b}{c\bar{z}+d}$$

where $a, b, c, d \in \mathbb{R}$ *and* $ad - bc > 0$ *preserves the upper-half plane.*

Definition

All such transformations form the group $\mathbb{R}\mathrm{M\ddot{o}b}$ of real Möbius transformations. The multiciplication in this group is function composition, and the identity is:

$$f(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1}$$

$\mathbb{R}$Möb Preserves the Half-Plane

Task

Show that $\mathbb{R}M\ddot{o}b$ *preserves the upper-half plane.*

If $\Im(z) > 0$ *and* $ad - bc > 0$ *then* $\Im\left(\frac{az+b}{cz+d}\right) > 0$

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$$\Im\left(\frac{az+b}{cz+d}\right) = \Im\left(\frac{az+b}{cz+d} \cdot \frac{\overline{cz+d}}{\overline{cz+d}}\right)$$

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$$\begin{aligned} \Im\left(\frac{az+b}{cz+d}\right) &= \Im\left(\frac{az+b}{cz+d} \cdot \frac{\overline{cz+d}}{\overline{cz+d}}\right) \\ &= \Im\left(\frac{(az+b)(\overline{cz+d})}{|cz+d|^2}\right) \end{aligned}$$

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\Im\left(\frac{az+b}{cz+d}\right) &= \Im\left(\frac{az+b}{cz+d}\cdot\frac{\overline{cz+d}}{\overline{cz+d}}\right)\\
&= \Im\left(\frac{(az+b)(\overline{cz+d})}{|cz+d|^2}\right)\\
&= \Im\left(\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2}\right)
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We have $(az+b)(c\overline{z}+d)$

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We have $(az+b)(c\overline{z}+d) = ac|z|^2 + adz + bc\overline{z} + bd$

$\mathbb{R}$Möb Preserves the Half-Plane

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\end{aligned}$$

We have $(az+b)(c\overline{z}+d) = ac|z|^2 + adz + bc\overline{z} + bd =$
$(ac|z|^2 + bd + (ad+bc)\Re(z)) + (ad-bc)i\Im(z)$

$\mathbb{R}$Möb is a Group

Task

Show that $\mathbb{R}M\ddot{o}b$ is closed under composition: $f(z), g(z) \in \mathbb{R}M\ddot{o}b \Rightarrow f(g(z)) \in \mathbb{R}M\ddot{o}b$

We now check:

$\mathbb{R}$Möb is a Group

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$f(g(z))$

We now check: $(\alpha a + \beta c)(\gamma b + \delta d) - (\alpha b + \beta d)(\gamma a + \delta c)$

$\mathbb{R}$Möb is a Group

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$$f(g(z)) \quad = \quad f\left(\frac{az+b}{cz+d}\right)$$

We now check: $(\alpha a + \beta c)(\gamma b + \delta d) - (\alpha b + \beta d)(\gamma a + \delta c) =$
$bc\beta\gamma - ad\beta\gamma - bc\alpha\delta + ad\alpha\delta$