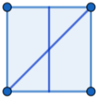
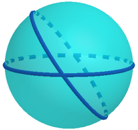
MAT 402: Classical Geometry

Groups

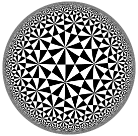


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

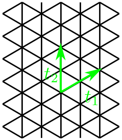
Spherical



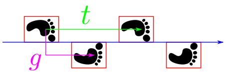
Hyperbolic



Tilings



Friezes

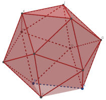


Trigonometry

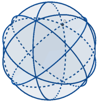
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

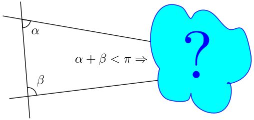
Platonic Solids

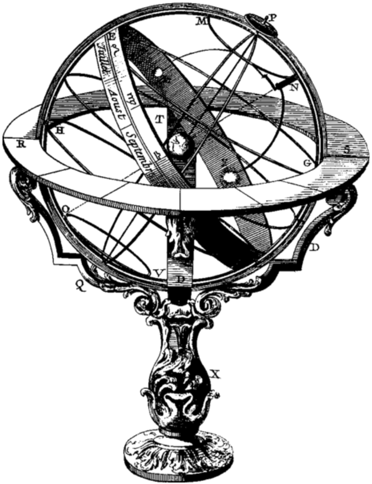


Coxeter



Parallels





From La Encyclopédie (1772) by Diderot and d'Alembert.

Welcome to continuous geometry. Questions? Comments?

MAT 402: Monday November 2nd 2020

Learning Objectives:

- Compare and contrast Euclidean and Spherical geometry.
- Prove analogues of theorems in planar geometry on the sphere.

Spherical Geometry

Definition

Spherical geometry is $(S^2 : O(3))$ where $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and $O(3)$ is the orthogonal group. (Alternatively, $O(3)$ is the isometries of $\mathbb{R}^3$ which send the origin to itself.)

Question

What is the difference between $SO(3)$ and $O(3)$?

Geodesics

Definition

A geodesic is a length minimizing path.

Question

What are the geodesics in standard Euclidean geometry?
How would you prove that they are length minimizing?

Polygonal Chains

Definition

A polygonal chain from p to q is a collection of distinct points $\mathcal{P} = \{P_0, P_1, \ldots, P_n\}$ with $p = P_0$ and $q = P_n$ joined by line segments $[P_i, P_{i+1}]$ for $i = 0, \ldots, n-1$.
The length of $\mathcal{P}$ is $||\mathcal{P}|| = \sum_{i=0}^{n-1} d(P_i, P_{i+1})$.

Task

Use the triangle inequality to show that the shortest polygonal chain between two points in $\mathbb{R}^2$ is a straight line.

Geodesics

Definition

A geodesic is a length minimizing path.

Question

What are the geodesics in standard Euclidean geometry?

Geodesics

Definition

A great circle is $C = \mathbb{S}^2 \cap \Pi$ where Π is a plane passing through $(0, 0, 0)$.

Theorem

The geodesics of $\mathbb{S}^2$ are great circles. (For example, longitudes or the equator.)

Distance

Definition

The distance between a pair of points $u, v \in \mathsf{S}^2$ is the angle between their vectors:

$$d(u, v) = \cos^{-1}(\vec{u} \cdot \vec{v})$$

Task

What are the spherical distances between $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$?

Pythagoras

Theorem (6.5.2 p. 120)

If ABC is a spherical right angled triangle with side-lengths $\{a, b, c\}$ and a right angle at C then:

$$\cos(c) = \cos(a)\cos(b)$$