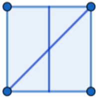
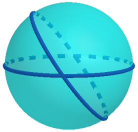
MAT 402: Classical Geometry

Groups

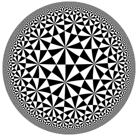


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

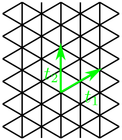
Spherical



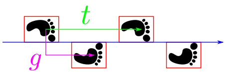
Hyperbolic



Tilings



Friezes

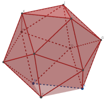


Trigonometry

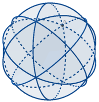
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

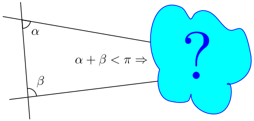
Platonic Solids

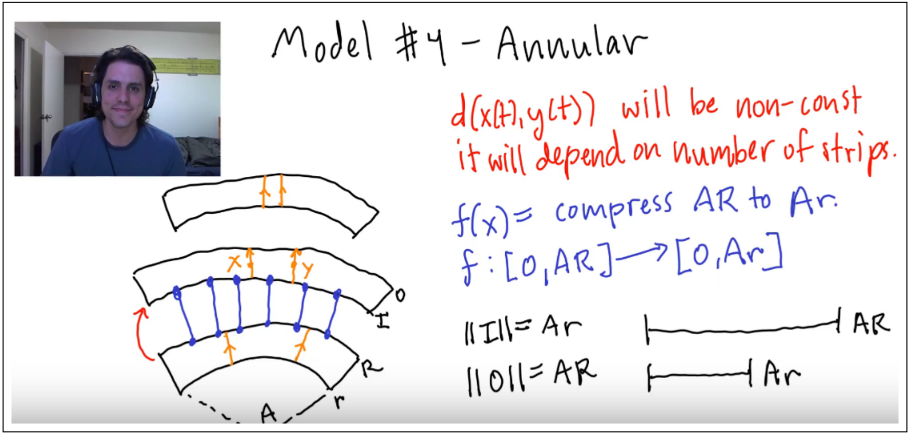


Coxeter



Parallels





Model #4 extended to have zero line Saturday @ 11:59am.
Questions? Comments?

MAT 402: Friday November 6th 2020

Learning Objectives:

- Prove analogues of theorems in planar geometry on the sphere.
- Calculate the area of spherical biangles and triangles.

Pythagoras

Theorem (6.5.2 p. 120)

If ABC is a spherical right angled triangle in $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ with side-lengths $\{a, b, c\}$ and a right angle at C then:

$$\cos(c) = \cos(a)\cos(b)$$

Task

How can we generalize this to $\mathbb{S}^2(R)$ a sphere of radius R?

Pythagoras on Spheres to Pythagoras on the Plane

Task

Given that $\cos(c/R) = \cos(a/R)\cos(b/R)$ *in a right angled triangle on* $\mathbb{S}^2(R)$*, how can we recover the usual Pythagorean theorem* $c^2 = a^2 + b^2$ *in the plane* $\mathbb{R}^2$*?*

Biangles

Definition (6.1)

A biangle S_α is a polygon on a sphere formed by two great circles that meet at angle α. The area of a biangle is $Area(S_\alpha) = 2\alpha$.

Task

What is the area of $\mathbb{S}^2$?

Girard's Theorem

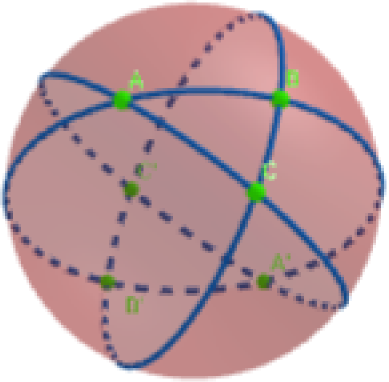
Theorem (6.4.5 Girard $\sim$ 1600 AD)

The area of a spherical triangle with angles $\{\alpha, \beta, \gamma\}$ is $S_{ABC} = \alpha + \beta + \gamma - \pi$.

Task

What is the area of the triangle with angles $\{\pi/2, \pi/2, \pi/2\}$?
Which tiling of the sphere do we get from it?

Girard's Theorem



https://www.geogebra.org/m/cqekqfrw

Covering with Biangles

Definition

We write B_{BAC} for the biangle at angle A with edges AB and AC.

Definition

The anti-podal triangle $A'B'C'$ is formed by extending the edges of ABC.

Task

Which biangles contain the triangle ABC? Does their union cover the sphere?

The Proof of Girard's Theorem

$$
\begin{aligned}
4\pi &= B_{BAC} + B_{ACB} + B_{CBA} + B_{B'A'C'} + B_{A'C'B'} + B_{C'B'A'} \\
&\quad -2Area(ABC) - 2Area(A'B'C') \\
&= B_{BAC} + B_{ACB} + B_{CBA} + B_{B'A'C'} + B_{A'C'B'} + B_{C'B'A'} - 4Area(ABC) \\
&= 2\alpha + 2\gamma + 2\beta + 2\alpha + 2\gamma + 2\beta - 4Area(ABC) \\
&= 4(\alpha + \beta + \gamma) - 4Area(ABC)
\end{aligned}
$$

We obtain: $Area(ABC) = \alpha + \beta + \gamma - \pi$.