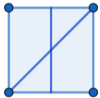
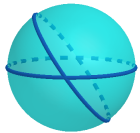
MAT 402: Classical Geometry

Groups

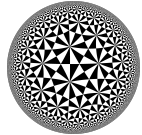


$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$

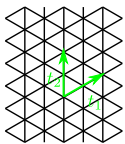
Spherical



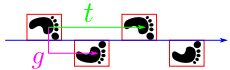
Hyperbolic



Tilings



Friezes

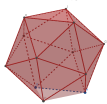


Trigonometry

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

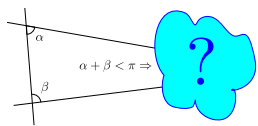
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

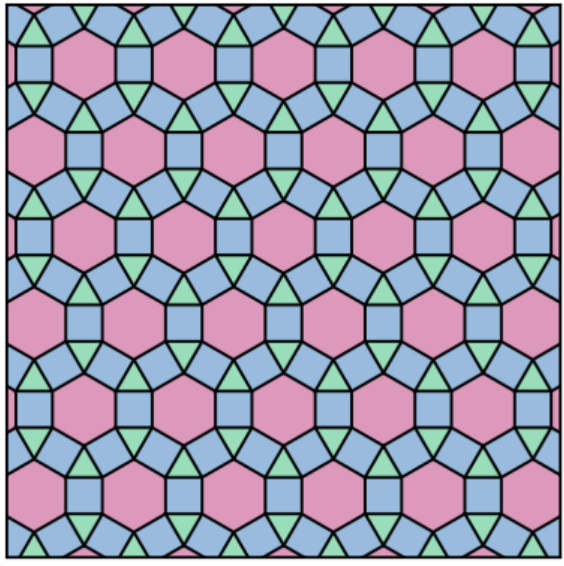
Platonic Solids



Coxeter

Parallels





Small Rhombitrihexagonal Tiling by R. A. Nonenmacher

Does Homework #3 seem shorter? Questions?

MAT 402: Monday October 26th 2020

Learning Objectives:

- Classify one-sided tilings.
- Distinguish regular and semi-regular tilings.
- Produce examples of tilings.

Fedorov's Theorem

Definition

If $G \subset \mathrm{Symm}^+(\mathbb{R}^2)$ then the tiling geometry $(T_0 : G)$ is called one-sided.

Theorem (Fedorov, 1891 : Book 4.5.3 Page 91)

There are exactly five different one-sided tiling geometries of the plane $\mathbb{R}^2$.

Rotation Lemma

Task

Suppose that $R_A(\theta)$ *and* $R_B(\phi)$ *are two rotations.*
Show that: $R_A(\theta)R_B(\varphi)R_A^{-1}(\theta)R_B^{-1}(\varphi)$ *is a translation.*

Translation Lemma

Definition

If $G \subset \mathrm{Symm}^+(\mathbb{R}^2)$ then G consists of rotations and translations.
Let $G_T \subseteq G$ be the subgroup of translations.

Task

Show that if $(T_0 : G)$ is a one-sided tiling geometry, then G_T is generated by two non-parallel translations $T_{\vec{x}}$ and $T_{\vec{y}}$.

Argue by contradiction:
If there are no translations, then...

If all the translations are parallel then...

Fedorov's Theorem

Theorem

Consider a one-sided tiling geometry $(T_0 : G)$. Let $R \subset \mathbb{R}^2$ be the set of centers of rotations. Let Δ be a minimal triangle contained in R. There are only finitely many possible congruence classes for Δ.

Task

Find all the $a \leq b \leq c$ such that:

$$\frac{\pi}{a} + \frac{\pi}{b} + \frac{\pi}{c} = \pi \iff \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

Produce a Tiling

Lemma

We have that $(a, b, c) = (2, 4, 4), (2, 3, 6), (3, 3, 3)$.
Produce a one-sided tiling for each of these solution.

Semi-Regular Tilings

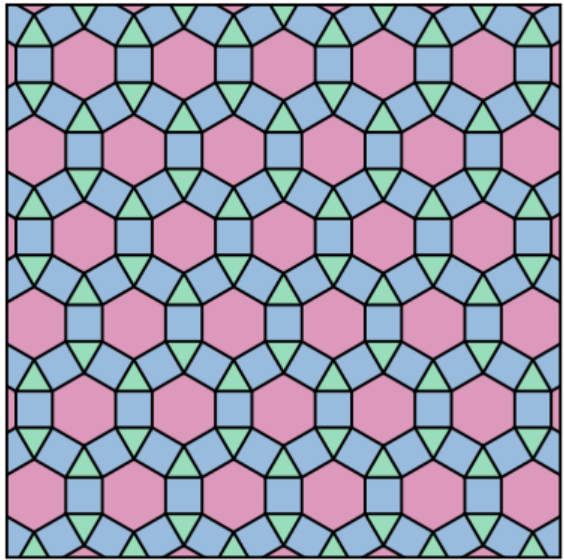
Definition

A regular tiling is a tiling of the plane by regular n-gons. A semi-regular tiling has the same collection of regular n-gons $\{n_1, n_2, \ldots, n_k\}$ surrounding it.

Task

Produce a regular tiling, and a semi-regular tiling.

Which Semi-Regular Tiling is This?



Small Rhombitrihexagonal Tiling by R. A. Nonenmacher.

Classifying Semi-Regular Tilings

Question

What is the internal vertex angle of a regular n-gon?

Task

If a semi-regular tiling has n-gons $\{n_1, \ldots, n_k\}$ *then what must equation must the numbers* $\{n_1, \ldots, n_k\}$ *satisfy to completely surround a vertex?*