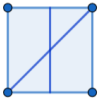
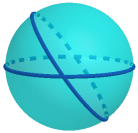
MAT 402: Classical Geometry

Groups

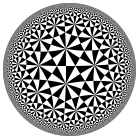


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

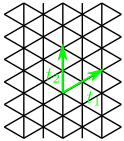
Spherical



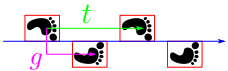
Hyperbolic



Tilings



Friezes

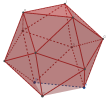


Trigonometry

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

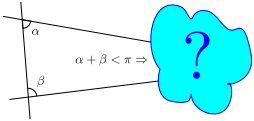
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

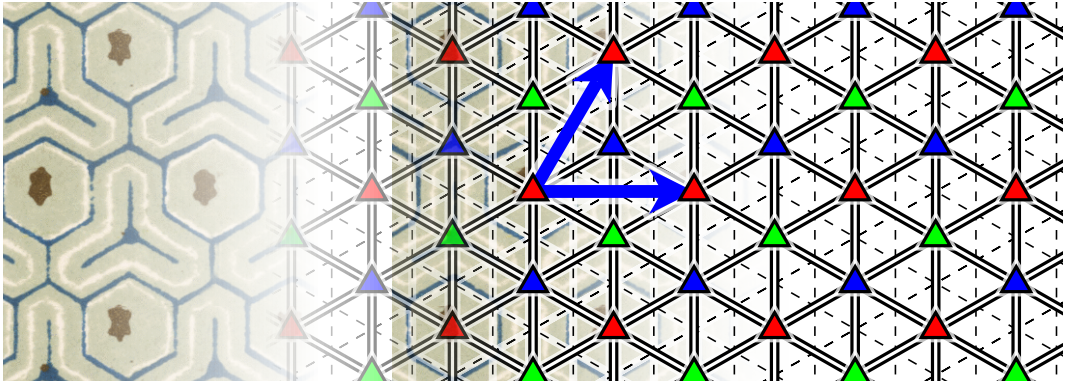
Platonic Solids



Coxeter

Parallels





Graphic by Martin von Gagern (http://www.morenaments.de/)

Questions? How are things going? Was the model super-duper hard?

MAT 402: Friday October 23th 2020

Learning Objectives:

- Distinguish one-sided and two-sided tilings.
- Produce examples of tilings.
- Create arguments about tilings.

Wallpaper Tilings

Definition

A tiling is *regular* if there is a group $G \subset \mathrm{Symm}(\mathbb{R}^2)$ and a bounded polygonal tile T_0 such that:

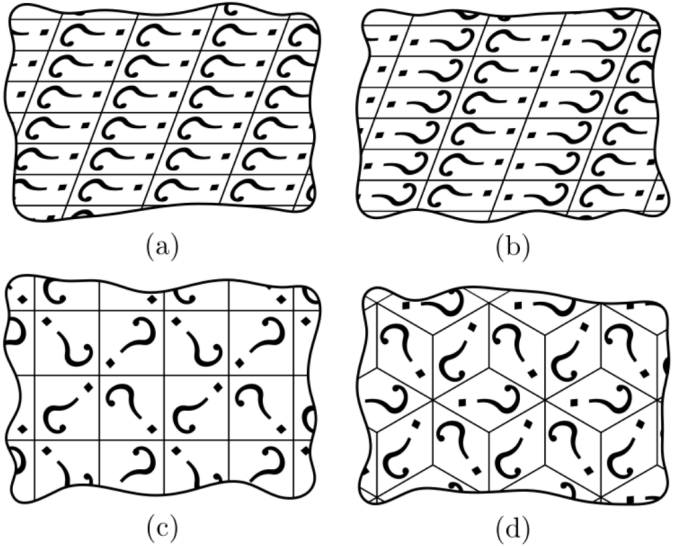
- G acts discretely on $\mathbb{R}^2$
- The orbit of T_0 fills the plane.

$$\mathbb{R}^2 = \bigcup_{g \in G} T_0 g$$

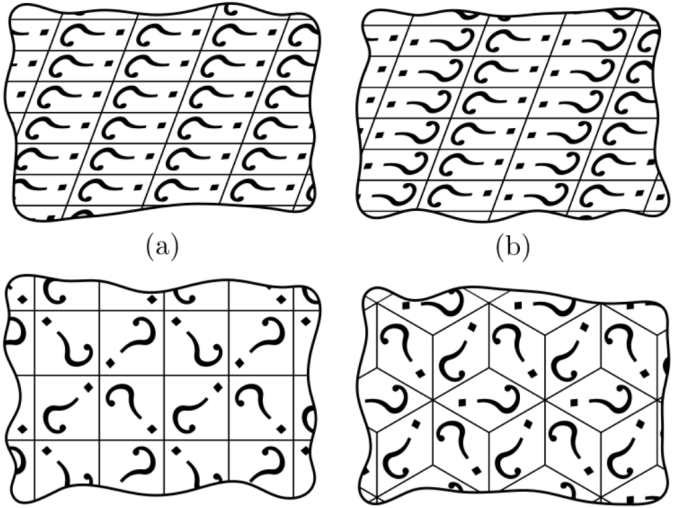
- The group action is transitive: $T_0 g = T_0 h \Rightarrow g = h$

We say $(\mathbb{R}^2 : G)$ is a *Fedorov geometry*, or *tiling geometry*.

Examples of Tilings (from Sossinsky's Geometries)



Which Tiling has $G = \mathbb{Z}^2$?



Create a Tiling

Task

Create a tiling with symmetry group not equal to $\mathbb{Z}^2$.

Create a Tiling à la Wythoff

Task

Create a tiling generated by reflections.

Fedorov's Theorem

Definition

If $G \subset \mathrm{Symm}^+(\mathbb{R}^2)$ then the tiling geometry $(T_0 : G)$ is called one-sided.

Theorem (Fedorov, 1891 : Book 4.5.3 Page 91)

There are exactly five different one-sided tiling geometries of the plane $\mathbb{R}^2$.

Rotation Lemma

Task

Suppose that $R_A(\theta)$ *and* $R_B(\phi)$ *are two rotations.*
Show that: $R_A(\theta)R_B(\varphi)R_A^{-1}(\theta)R_B^{-1}(\varphi)$ *is a translation.*

Translation Lemma

Definition

If $G \subset \mathrm{Symm}^+(\mathbb{R}^2)$ then G consists of rotations and translations.
Let $G_T \subseteq G$ be the subgroup of translations.

Task

Show that if $(T_0 : G)$ is a one-sided tiling geometry, then G_T is generated by two non-parallel translations $T_{\vec{x}}$ and $T_{\vec{y}}$.

Argue by contradiction:
If there are no translations, then...

If all the translations are parallel then...