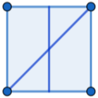
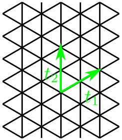
# MAT 402: Classical Geometry

## Groups

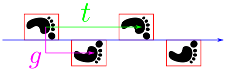


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

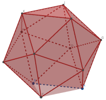
## Tilings



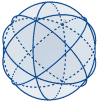

## Friezes



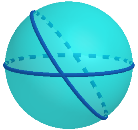

## Platonic Solids



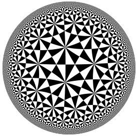
## Coxeter



## Spherical



## Hyperbolic

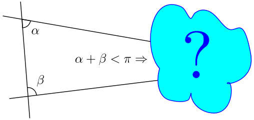


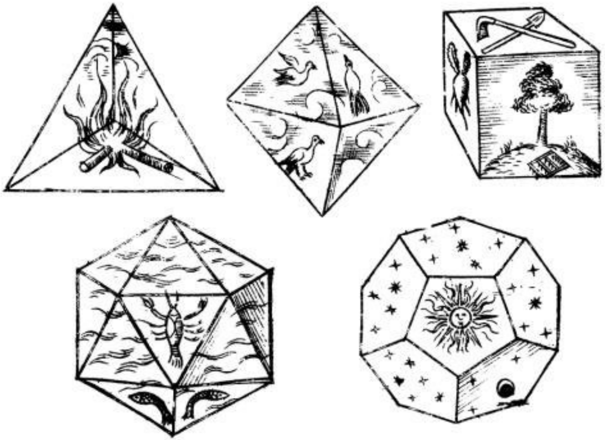
## Trigonometry

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

## Parallels





Kepler's drawings of the Platonic solids

**Any question about the proof so far? Homework? Comments?**

# From Last Lecture

**Theorem**

*If* $G^+ \subset \mathrm{SO}(3)$ *is a finite and* $A$ *contains a point from each orbit, then:*

$$2 - \frac{2}{|G^+|} = \sum_{a \in A} \left(1 - \frac{1}{|\operatorname{St}(a)|}\right)$$

# The Case $|A| = 2$

Let $A = \{a_1, a_2\}$. We have:

$$2 - \frac{2}{|G^+|} = \left(1 - \frac{1}{|\operatorname{St}(a_1)|}\right) + \left(1 - \frac{1}{|\operatorname{St}(a_2)|}\right) \iff \frac{2}{|G^+|} = \frac{1}{|\operatorname{St}(a_1)|} + \frac{1}{|\operatorname{St}(a_2)|}$$

Multiplying through by $|G^+|$ we get:

$$2 = \frac{|G^+|}{|\operatorname{St}(a_1)|} + \frac{|G^+|}{|\operatorname{St}(a_2)|} = |\operatorname{Orb}(a_1)| + |\operatorname{Orb}(a_2)|$$

We can conclude: $|\operatorname{Orb}(a_1)| = |\operatorname{Orb}(a_2)| = 1$.

**Task**

*What symmetric objects have this property?*
*What is the corresponding symmetry group?*

# The Case $|A| = 3$

Let $A = \{a_1, a_2, a_3\}$. We have:

$$\begin{aligned} 2 - \frac{2}{|G^+|} &= \left(1 - \frac{1}{|\operatorname{St}(a_1)|}\right) + \left(1 - \frac{1}{|\operatorname{St}(a_2)|}\right) + \left(1 - \frac{1}{|\operatorname{St}(a_3)|}\right) \\ \frac{2}{|G^+|} &= \frac{1}{|\operatorname{St}(a_1)|} + \frac{1}{|\operatorname{St}(a_2)|} + \frac{1}{|\operatorname{St}(a_3)|} - 1 \end{aligned}$$

**Task (2 min)**

*Check the tetrahedral case:*
$|\operatorname{St}(a_1)| = |\operatorname{St}(a_2)| = 3$, *and* $|\operatorname{St}(a_3)| = 2$, *with* $|G^+| = |\operatorname{Symm}^+(\Delta)| = 12$

**Task (2 min)**

*Check the cubical case:*
$|\operatorname{St}(a_1)| = 2$, $|\operatorname{St}(a_2)| = 3$, *and* $|\operatorname{St}(a_3)| = 4$, *with* $|G^+| = |\operatorname{Symm}^+(I^3)| = 24$

# The Case $|A| = 3$ continued

Notice that because the left hand side $\frac{2}{|G^+|} > 0$ is positive we have:

$$\frac{1}{|\operatorname{St}(a_1)|} + \frac{1}{|\operatorname{St}(a_2)|} + \frac{1}{|\operatorname{St}(a_3)|} > 1$$

**Task (2 min)**

*Show (arithmetically) that* $2 \in \{\operatorname{St}(a_1), \operatorname{St}(a_2), \operatorname{St}(a_3)\}$.

**Task (15 min)**

*Let* $|\operatorname{St}(a_1)| = X$, $|\operatorname{St}(a_2)| = Y$ *and* $|\operatorname{St}(a_3)| = Z$.
*Find all possible sizes of* $X \leq Y \leq Z$ *such that the inequality holds.*

# Group Orders

We have that $\{2, 2, n\}$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, and $\{2, 3, 5\}$ are the only possible sizes of stabilizers when $|A| = 3$.

## Task (10 min)

*Use the fundamental equation to determine the order of the groups.*

$$2 - \frac{2}{|G^+|} = \sum_{a \in A} \left(1 - \frac{1}{|\operatorname{St}(a)|}\right)$$

# The Final Classification

| $\mathrm{St}(a_1)$ | $\mathrm{St}(a_2)$ | $\mathrm{St}(a_3)$ | $G^+$ | Symmetric Object |
|---|---|---|---|---|
| $n$ | $n$ | | $n$ | $n$-gonal pyramid |
| 2 | 2 | $n$ | $2n$ | $n$-gonal bi-pyramid |
| 2 | 3 | 3 | 12 | Tetrahedron |
| 2 | 3 | 4 | 24 | Cube/Octohedron |
| 2 | 3 | 5 | 60 | Dodecahedron/Icosahedron |