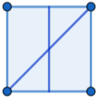
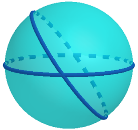
MAT 402: Classical Geometry

Groups

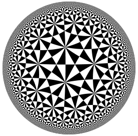


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

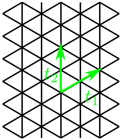
Spherical



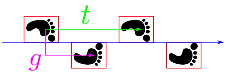
Hyperbolic



Tilings



Friezes

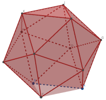


Trigonometry

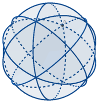
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

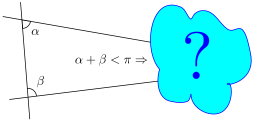
Platonic Solids



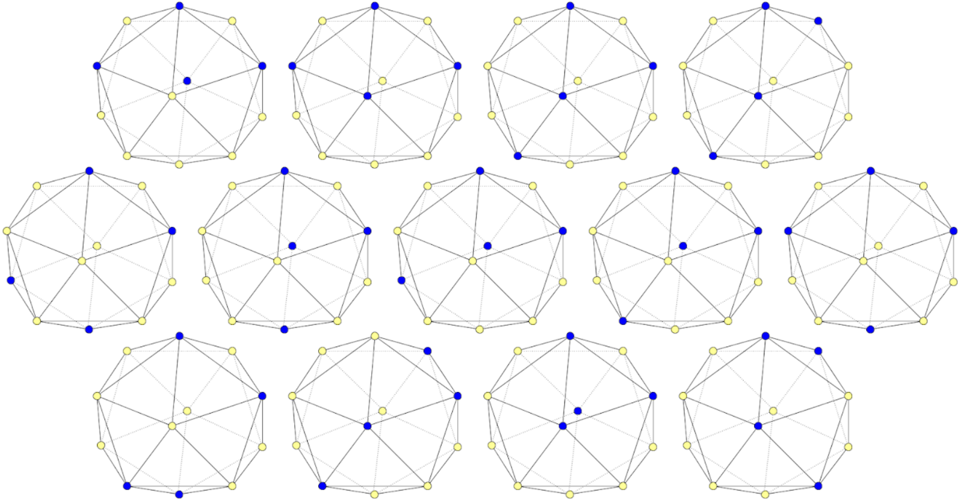
Coxeter



Parallels



News Flash: Feedback #1 is out! Quest 1 is available.



Platonic Perplexer by David Nacin

MAT 402: Monday September 28th 2020

Learning Objectives:

- Relate the order of a rotational pole to the size of $G^+ \subset SO(3)$
- Explain the geometric significance of orbits and stabilizers in $\mathbb{R}^3$.

Rotational Symmetries of the Cube

Task (10 min)

There are 24 rotational symmetries of the cube.
What are the orders of these symmetries? Provide examples.

To play with the cube: https://www.geogebra.org/m/Fdb6Mq6v

Rotational Symmetries of the Cube

Question (10 min)

Suppose a pole p of a rotational symmetry of the cube has order k.
How does the size of the orbit of p relate to k?

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Definition

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We write $H + g$ if the operation on G is addition.

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Task

Find all distinct cosets of $H = \{0, 2, 4\} \subset \mathbb{Z}_6$.

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If H is a subgroup of G then: (i) all H cosets have the same cardinality and (ii) any two H cosets are equal or disjoint.

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(2) If $g \in Hg_i \cap Hg_j \neq \emptyset$ then $g = h_1 g_i = h_2 g_j$ thus, $g_j = h_2^{-1} h_1 g_i \overset{\star}{=} hg_i$.
We then have $Hg_j = H(hg_i) \overset{\star}{=} Hg_i$.

The equalities $\star$ hold because $\boxed{}$. □

Stabilizer Cosets

Lemma

If p *is a point and* $\mathrm{St}(p)$ *is its stabilizer subgroup with cosets* $\{\mathrm{St}(p)g_1, \ldots, \mathrm{St}(p)g_q\}$ *then: (i)* $p\,\mathrm{St}(p)g_i$ *is unique, (ii)* $p\,\mathrm{St}(p)g_i$ *is distinct for each* $i \in \{1, \ldots, q\}$.

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It follows $\mathrm{St}(p) = \mathrm{St}(p)g_jg_i^{-1}$ because $\mathrm{St}(p)$ is a subgroup.
Therefore, multiplying by g_i, we get that the cosets are identical:

$$\mathrm{St}(p)g_i = \mathrm{St}(p)g_j$$

□

Fixed Point Counting

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Task (10 min)

What does this tell us about the poles of rotation of the cube?