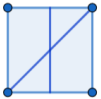
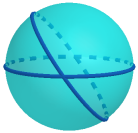
MAT 402: Classical Geometry

Groups

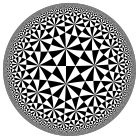


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

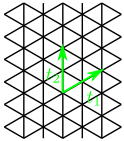
Spherical



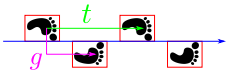
Hyperbolic



Tilings



Friezes

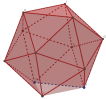


Trigonometry

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

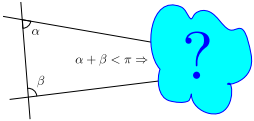
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

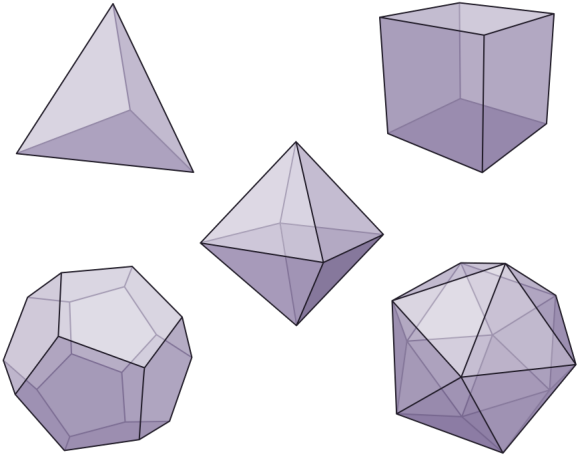
Platonic Solids



Coxeter

Parallels





Any question about the homework? Comments?

MAT 402: Friday October 2nd 2020

Learning Objectives:

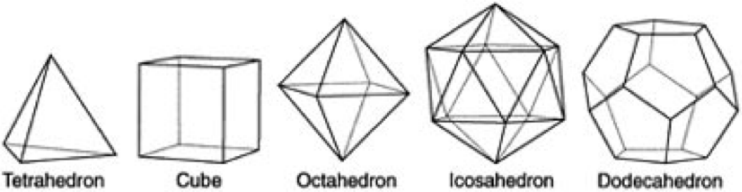
- Classify the finite subgroups of $SO(3)$
- Determine the number of orbits for a general finite subgroup of $SO(3)$.

The Classification Theorem

Theorem (Archimedes)

Suppose that $G^+ \subset SO(3)$ is finite and non-trivial.
G^+ is isomorphic to one of the following:

- $\mathbb{Z}_n$ *for* $n \geq 2$.
- $\mathbb{D}_n$ *for* $n \geq 2$.
- $\mathrm{Symm}^+(\Delta)$ *from the tetrahedron*
- $\mathrm{Symm}^+(I^3)$ *from the cube/octohedron*
- $\mathrm{Symm}^+(\mathrm{Dod})$ *from dodecahedron/icosahedron*



The Class Formula

Theorem ("Orbit-Stabilizer" for Poles)

If p is a pole of order k and $G^+ \subset \mathrm{SO}(3)$ then $|\operatorname{Orb}(p)| = |G^+|/k = |G^+|/|\operatorname{St}(a)|$.

Theorem (The Class Formula)

If G^+ acts on a set F we can decompose it as a collection of orbits: $F = \bigcup_{a \in A} \operatorname{Orb}(a)$ where A contains an element from each orbit. We obtain:

$$|F| = \sum_{a \in A} |\operatorname{Orb}(a)| = \sum_{a \in A} \frac{|G^+|}{|\operatorname{St}(a)|}$$

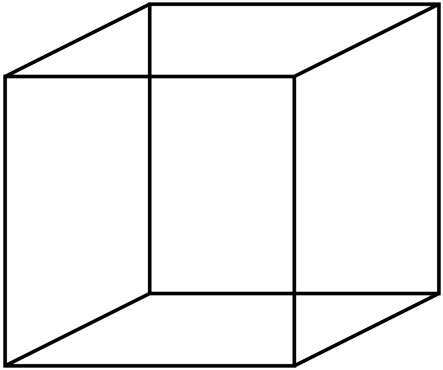
Task (See next slide)

Check this formula for the cube.

The Cube

Task (10 min)

How does this apply to the cube? $|F| = \sum_{a \in A} |\operatorname{Orb}(a)| = \sum_{a \in A} \frac{|G^+|}{|\operatorname{St}(a)|}$



The Geometric Relation

Theorem

Let G^+ be a finite subgroup of $SO(3)$ acting on the sphere $\mathbb{S}^2$ and $F = \{x \mid \exists g \in G^+ \setminus \{e\} : gx = x\}$ be the set of all the points fixed by non-trivial elements of G^+. Let $A \subset F$ be a set containing exactly one point from each orbit of G^+ acting on F. We have the following identity:

$$|F| = |G^+||A| - 2(|G^+| - 1)$$

Some observations about this identity:

- $|G^+||A|$ will dramatically overcount fixed points.
- Each element of $G^+ \setminus \{e\}$ has an axis of rotation with two poles. This gives the term $2(|G^+| - 1)$ poles (which will be in $|F|$).

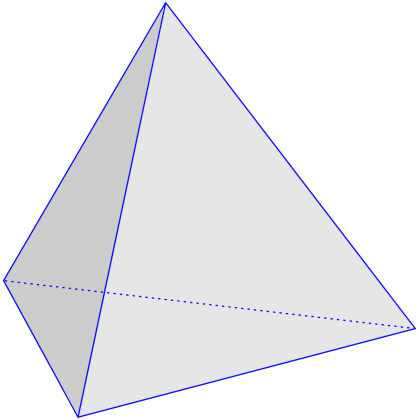
Task (See next slide)

Check this identity for the tetrahedron.

The Tetrahedron

Task (10 min)

How does this apply to the tetrahedron? $|F| = |G^+||A| - 2(|G^+| - 1)$



Marriage of the Class Formula and the Geometric Relation

We have the following statements about $|F|$.

$$|F| = |G^+||A| - 2(|G^+| - 1) \quad \text{and} \quad |F| = \sum_{a \in A} |\operatorname{Orb}(a)| = \sum_{a \in A} \frac{|G^+|}{|\operatorname{St}(a)|}$$

Equating these, and dividing through by $|G^+|$ gives:

$$|A| - 2 + \frac{2}{|G^+|} = \sum_{a \in A} \frac{1}{|\operatorname{St}(a)|}$$

Note that $|A| = \sum_{a \in A} 1$. So, we can re-arrange and obtain:

$$-2 + \frac{2}{|G^+|} = \sum_{a \in A} \left(\frac{1}{|\operatorname{St}(a)|} - 1 \right) \iff \boxed{2 - \frac{2}{|G^+|} = \sum_{a \in A} \left(1 - \frac{1}{|\operatorname{St}(a)|} \right)}$$

The Diophantine Equation for Finite Subgroups

Theorem

If $G^+ \subset \mathrm{SO}(3)$ is a finite and A contains a point from each orbit, then:

$$2 - \frac{2}{|G^+|} = \sum_{a \in A} \left(1 - \frac{1}{|\mathrm{St}(a)|}\right)$$

We are going to use this relationship to determine all possible finite subgroups.

Task (2 min)

If $|G^+| > 1$ then what bounds must the left hand side satisfy?

Task (2 min)

What bounds must the summands of the right hand side satisfy?

The Enumeration of Cases

Question (5 min)

Given that LHS $\in [1, 2)$ *and* $\frac{1}{2} \leq 1 - \frac{1}{|\operatorname{St}(a)|}$, *how many cases are there?*