Absolute Constants and Hyperbolic Trigonometry

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- ▶ Isomorphism of the Cayley-Klein and Poincaré disk models
- ▶ Isomorphism of the two Poincaré models
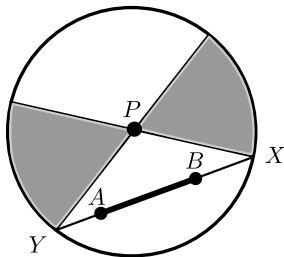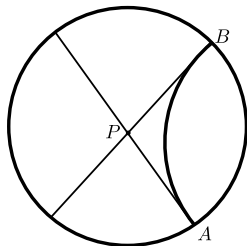- ▶ Hyperbolic trigonometry functions
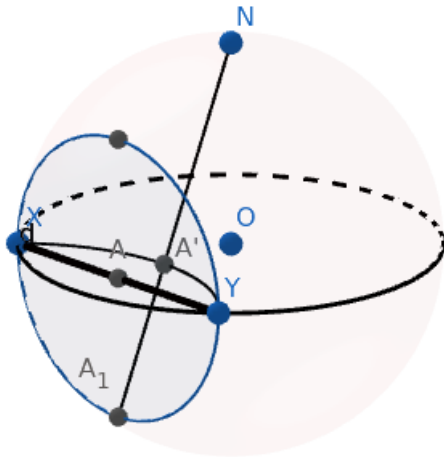- ▶ Absolute constants

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Cayley and Poincaré

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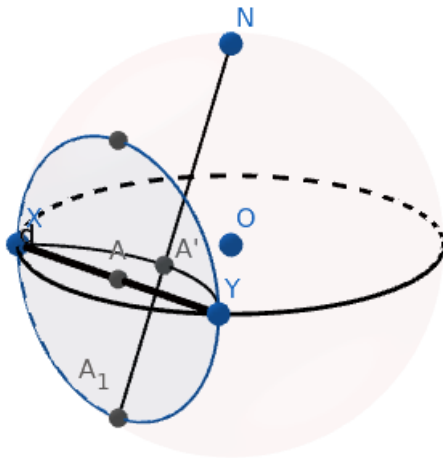**TORONTO**
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https://www.geogebra.org/3d/gzzatbga

Cayley and Poincaré

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The chord XY is a geodesic in the Cayley-Klein model.

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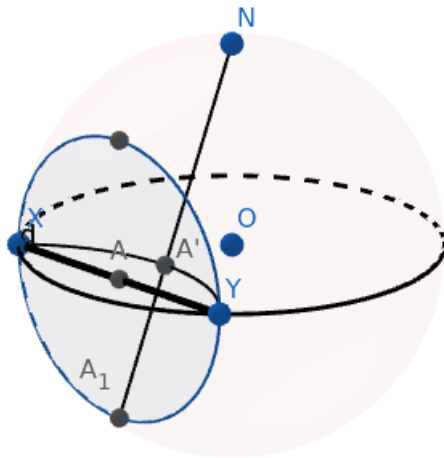
Cayley and Poincaré

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The chord XY is a geodesic in the Cayley-Klein model.

The arc $XA'Y$ is the corresponding geodesic in the Poincaré model.

https://www.geogebra.org/3d/gzzatbga

The Poincaré Disk and Half-Plane

Consider the map $\Omega : \mathbb{D}^2 \to \mathbb{C}_+$.

$$\Omega(z) = i \cdot \frac{1+z}{1-z}$$

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$\mathbb{D}^2$

$\frac{1}{2}i$

0 $\frac{1}{2}$
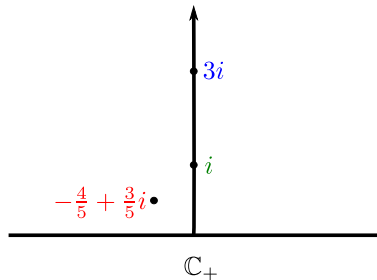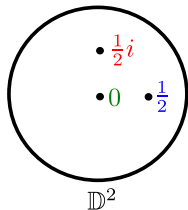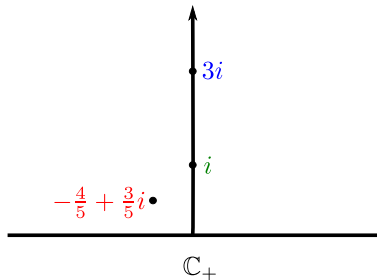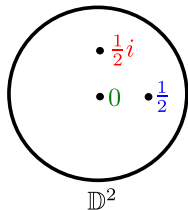
$3i$

i

$-\frac{4}{5} + \frac{3}{5}i$

$\mathbb{C}_+$

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The Poincaré Disk and Half-Plane

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It induces an isomorphism of geometries:

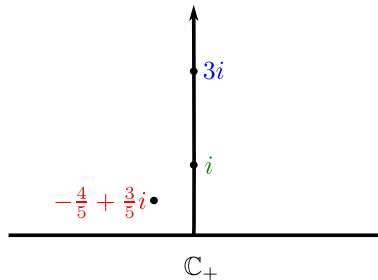## The Poincaré Disk and Half-Plane

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$$(\mathbb{H}^2, \mathcal{M}) \simeq (\mathbb{C}_+, \mathbb{R}\text{Möb})$$

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Corollary (10.2.3)

The three models of hyperbolic geometry, namely the Poincaré disk and half-plane models and the Cayley-Klein model, are isomorphic as geometries.

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Hyperbolic Trigonometry

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Absolute Constants

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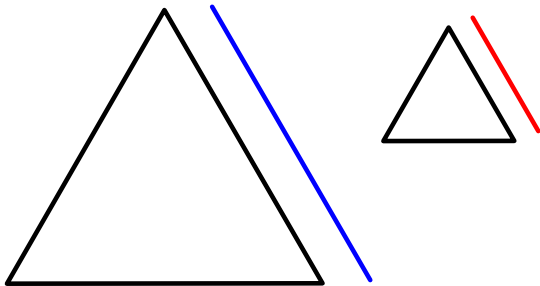Similarity of figures implies there is no natural reference point.

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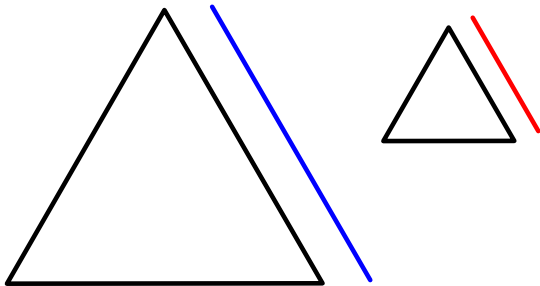
In hyperbolic geometry, there are natural units!

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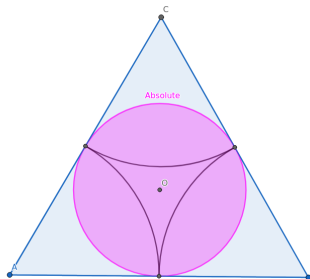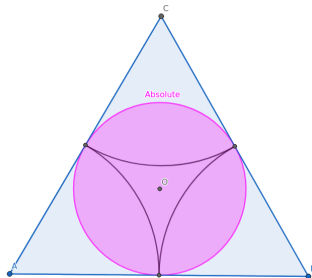
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Definition

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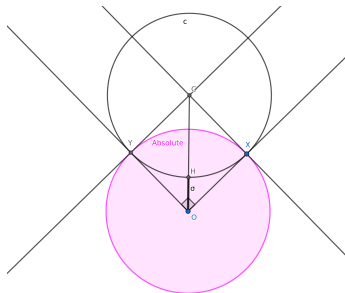
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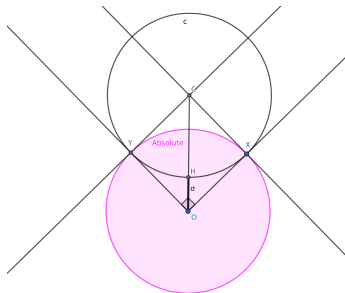
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