Fundamental inequalities

Here are the most basic inequalities, for ease of reference. These notes follow Paul Zeitz's "The Art and Craft of Problem Solving", with certain constants changed.

Theorem. *If a, b, c are real numbers, we have the following:*

1. *If $a \leq b$ and $b \leq c$, then $a \leq c$.*
2. *If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.*
3. *If $a \leq b$ and $0 \leq c$ then $ac \leq bc$.*
4. *If $a \leq b$ and $c \leq 0$ then $ac \geq bc$.*
5. *If $0 < a \leq b$ then $1/a \geq 1/b$.*

Example: There are no natural numbers m, n for which m^2 can be written in the form $n^2 + n + 1$.

Solution: If there were, then we'd have $n^2 < n^2 + n + 1 < n^2 + 2n + 1 = (n + 1)^2$, which says that the middle term (which is m^2 by hypothesis) is a square strictly between two consecutive squares! This is a very slick contradiction, so we're done.

1. Use the fundamentals of inequalities to prove the following:
   (a) Show that if $0 < a < b$, then $x^a < x^b$ for large values of x. How large is "large"?
   (b) If $b > 1$ and $a > 0$, prove that $x^a < b^x$ for large x. Again, how large?
2. All kinds of comparison questions, geometric optimization questions, and rate questions are just basic inequalities in disguise, and the trick is to un-disguise them. Sometimes it's useful to introduce a clever function to do all the work for you.
   (a) Which is bigger, 2013/2014 or 2014/2015?
   (b) Which is bigger, $\dfrac{10^{2014}+1}{10^{2013}+1}$ or $\dfrac{10^{2015}+1}{10^{2014}+1}$?
3. Other times it's useful to consider basic estimates (like $2^3 < 3^2$) and then group terms.
   (a) Which is bigger, 1000^{2000} or $(2000 \cdot 1999 \cdots 2 \cdot 1)$?
   (b) Which is bigger, 9^{10} or 10^9?
4. To test if $x \geq y$, consider testing if $x/y \geq 1$ or $x - y \geq 0$.
   (a) (CMO 1997) Prove that:
   $$\frac{1}{1999} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{1997}{1998} < \frac{1}{44}$$
   (b) (CMO 1973)
      i. Find a inequality equivalent to $x < 1/(4x)$ and $x < 0$.
      ii. What is the greatest integer that satisfies both $4x + 13 < 0$ and $x^2 + 3x > 16$?
      iii. Give a rational number between 11/24 and 6/13.

5. (CMO 1973) Find all real numbers which satisfy the equation $|x+3|-|x-1|=x+1$.
6. (CMO 1975) Let $\lfloor x \rfloor$ denote the largest integer less than x. Plot all points such that $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 4$.
7. One of the most basic inequalities is the triangle inequality, which says that for any real numbers x, y, we have

$$|x+y| \leq |x| + |y|$$

   Prove this fact by considering cases for the absolute value function.
8. The reverse triangle inequality says that $||x|-|y|| \leq |x+y|$. Prove this!
9. (CMO 1977) Let $0 < u < 1$ and define

$$u_1 = 1+u, \quad u_2 = \frac{1}{u_1}+u, \quad u_{n+1} = \frac{1}{u_n}+u$$

   Show that $u_n > 1$ for all $n \geq 1$.
10. (CMO 1992) For $x, y, z \geq 0$ show that:

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$$

   and determine when equality holds.

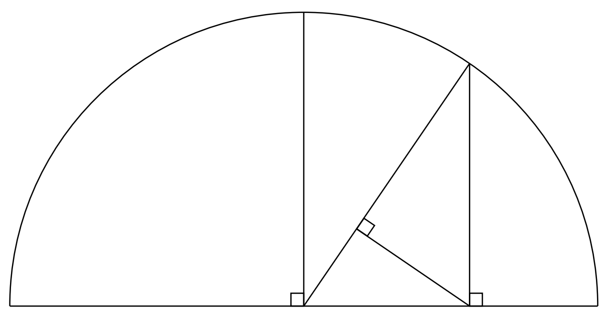
Intermediate inequalities

A basic but useful fact about real numbers is that their squares are always nonnegative. In other words, $x^2 \geq 0$ for all $x \in \mathbb{R}$, with equality if and only if $x = 0$. A nice consequence of this is an inequality between the arithmetic, geometric, and harmonic means of two positive numbers.

Theorem (Baby AMGMHM). *For any positive a and b, we have*

$$\frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$

1. Prove the baby AMGM and GMHM inequalities in two different ways:
   (a) Rearrange each inequality and look for a perfect square.
   (b) In this picture of a semicircle with diameter $a+b$, fill in the lengths of all the triangles.



   (c) Based on your two proofs, under what conditions does equality hold?

2. Now use what you know to prove that if a, b, c are positive numbers, then
   (a) $a^2 + b^2 + c^2 \geq ab + bc + ca$
   (b) $(a+b)(b+c)(c+a) \geq 8abc$
   (c) $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$
   (d) $ab + bc + ca \leq 1/3$ whenever $a + b + c = 1$

3. Show that $\cos^2 x + x \sin x < 2$ whenever $0 < x < \pi/2$.

4. If a, b, c and d are positive numbers with $(1+a)(1+b)(1+c)(1+d) = 16$, show that $abcd \leq 1$.

5. Prove that for all positive integers n, we have $n! \leq 2(n/2)^n$.

6. Prove that for any positive integer n, we have $\sqrt[n]{n} < 1 + \sqrt{2/n}$.

7. For $0 < a < b$ and $n \in \mathbb{N}$, show that $(n+1)(b-a)a^n < b^{n+1} - a^{n+1} < (n+1)(b-a)b^n$.

8. Prove that $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{999999}{1000000} < \frac{1}{1000}$. Hint: square both sides and pair up the factors.

9. (a) If a and b are nonzero real numbers, show that we have at least one of the following:

$$\left| \frac{a + \sqrt{a^2 + 2b^2}}{2b} \right| < 1 \quad \text{or} \quad \left| \frac{a - \sqrt{a^2 + 2b^2}}{2b} \right| < 1$$

   (b) Let $x_1, \ldots, x_n$ be numbers in $(0,1)$. Show that at least one of the following holds:

$$x_1 x_2 \cdots x_n \leq 2^{-n} \quad \text{or} \quad (1-x_1)(1-x_2)\cdots(1-x_n) \leq 2^{-n}$$