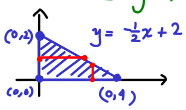
Changing Order of Integration (§5.4)

Fubini's Theorem:

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx$$

Ex: Write down both orders of integration for $f(x,y)$ on the triangle bounded by $(0,0)$ $(0,2)$ and $(4,0)$

Sketch the region



$y = -\frac{1}{2}x + 2 \Leftrightarrow y-2 = -\frac{1}{2}x$

$\Leftrightarrow -2y+4 = x$

$\iint f(x,y)\,dx\,dy$

$\int_0^2 \int_0^{-2y+4} f(x,y)\,dx\,dy$

$\iint f(x,y)\,dy\,dx$

$\int_0^4 \int_0^{-\frac{1}{2}x+2} f(x,y)\,dy\,dx$

Work from outside bound to inner bound

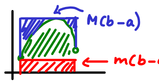
Min/Max Inequality

Recall, any cont. fxn on compact closed set S achieves minimum m and maximum M (The Extreme Value Theorem)

Let the area of S be $A(S) \Rightarrow mA(S) \le \iint_S f(x,y)\,dA \le MA(S)$

For one variable, $f: [a,b] \to \mathbb{R}$

$m(b-a) \le \int_a^b f(x)\,dy \le M(b-a)$



The real area is b/w the two boxes

Ex: Show $\frac{4\pi^2}{e} \le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{\sin(x+y)}\,dx\,dy \le 4\pi^2 e$

Estimate function

$-1 \le \sin(\theta) \le 1 \Rightarrow e^{-1} \le e^{\sin(\theta)} \le e^1 \Rightarrow \frac{1}{e} \le e^{\sin\theta} \le e$

Calculate the area of R

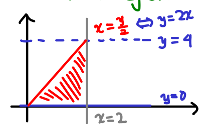
$A(R) = \underbrace{(\pi-(-\pi))}_{\text{height}}\underbrace{(\pi-(-\pi))}_{\text{width}} = (2\pi)(2\pi) = 4\pi^2$

Ex (§5.4 Q1c) Integrate $\int_0^4 \int_{\frac{y}{2}}^{2} e^{x^2}\,dx\,dy$ $\quad \int e^{x^2}dx = \int\left(\sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n\right)dx$

There is no simple anti-derr!

Change order of Integration

Sketch the region

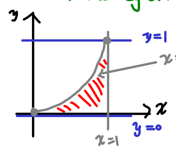


$\int_0^4 \int_{\frac{y}{2}}^{2} e^{x^2}\,dx\,dy = \int_0^2 \int_0^{2x} e^{x^2}\,dy\,dx = \int_0^2 \left[ye^{x^2}\right]_0^{2x} dx$

$= \int_0^2 \left[2xe^{x^2}\right] dx \quad \underset{u=x^2 \quad du = 2x\,dx}{\text{u-sub}} \Rightarrow \int_0^4 e^u\,du = e^4$

Ex: (§5.4 Q5) $\int_0^1 \int_{\sqrt{y}}^{1} e^{x^3}\,dx\,dy$ no simple anti-derr for e^{x^3}

Sketch the region



$\int_0^1 \int_{\sqrt{y}}^{1} e^{x^3}\,dx\,dy = \int_0^1 \int_0^{x^2} e^{x^3}\,dy\,dx$

$= \int_0^1 \left[ye^{x^3}\right]_0^{x^2} dx$

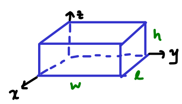
usub $u = x^3 \quad du\ 3x^2$ $\quad = \int_0^1 \left[x^2 e^{x^3}\right] dx$

$\Rightarrow \int_0^1 \frac{e^u}{3}\,du = \frac{e}{3}$

Triple Integrals

$\iiint_R f(x,y,z)\,dV$ ← volume

Ex: Find the volume of a box of height h, width w, length l



$V = \int_0^h \int_0^w \int_0^l 1\,dx\,dy\,dz = \int_0^h \int_0^w [x]_0^l\,dy\,dz$

$= \int_0^h \int_0^w l\,dy\,dz = \int_0^h [ly]_0^w\,dz = \int_0^h lw\,dz$

$= [lwz]_0^h = lwh$

Triple Integrals (§8.5)

Ex: Evaluate $\int_0^1 \int_0^2 \int_0^3 x^2\,dx\,dy\,dz$ (!) work inside to outside

Integrate in dx

$\int_0^1 \int_0^2 \int_0^3 x^2\,dx\,dy\,dz$

$= \int_0^1 \int_0^2 \left[\frac{1}{3}x^3\right]_0^3 dy\,dz$

Integrate in dy

$= \int_0^1 \int_0^2 9\,dy\,dz$

$= \int_0^1 [9y]_0^2\,dz$

Integrate in dz

$= \int_0^1 18\,dz$

$[18z]_0^1 = 18$

(?) What does it mean? $w = f(x,y,z)$ is in 4-D

Triple Integral is (hyper) volume

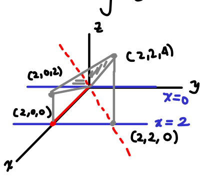
Ex: (§5.5 Q4) $\int_0^1 \int_0^1 \int_0^1 ye^{-xy}\,dx\,dy\,dz$

$= \int_0^1 \int_0^1 \left[-e^{-xy}\right]_0^1 dy\,dz = \int_0^1 \int_0^1 e^{-y} + e^0\,dy\,dz = \int_0^1 \int_0^1 1 - e^{-y}\,dy\,dz$

$= \int_0^1 [y + e^{-y}]_0^1\,dz = \int_0^1 1 + e^{-1} - \underbrace{(0 + e^{-0})}_{1}\,dz = \int_0^1 e^{-1}\,dz = e^{-1}$

Ex: Sketch the region of integration for $\int_0^2 \int_0^x \int_0^{x+y} f(x,y,z)\,dz\,dy\,dx$

We get:
$0 \le x \le 2$
$0 \le y \le x$
$0 \le z \le x+y$



Applications of Integration (§6.3)

Averages: $\text{Avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$ "Average value of $f(x)$ on the interval $[a,b]$"

$b-a$: Length$([a,b])$

In two-variables: $\text{Avg} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$

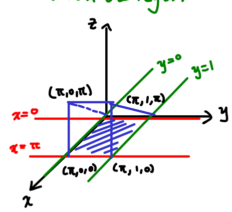
Ex: What is the average of $f(x,y) = 24$ over $R = [0,1]\times[0,2]$

$\text{Avg} = \frac{1}{\text{Area}([0,1]\times[0,2])} \int_0^1 \int_0^2 24\,dy\,dx = \frac{1}{(1-0)(2-0)} \int_0^1 [24y]_0^2\,dx$

$= \frac{1}{2}\int_0^1 48\,dx = \frac{1}{2}[48x]_0^1 = \frac{48}{2} = 24$

Ex: Evaluate $\iiint_R \sin(x)\,dV$ R is bounded by $0 \le x \le \pi$; $0 \le y \le 1$; $0 \le z \le x$

Sketch the region



Setup Integral

$\int_0^1 \int_0^{\pi} \int_0^{x} \sin(x)\,dz\,dx\,dy = \int_0^1 \int_0^{\pi} \left[z\sin(x)\right]_0^x dx\,dy$

$= \int_0^1 \int_0^{\pi} x\sin(x)\,dx\,dy$ ← IBP

$= \int_0^1 \left[\sin(x) - x\cos(x)\right]_0^{\pi} dy$

$= \int_0^1 [\sin(\pi) - \pi\cos(\pi)] - (\sin(0) - 0)\,dy$

$= \int_0^1 \pi\,dy = \pi$

$dv = dx\,dy\,dz = dy\,dx\,dz = dz\,dx\,dy$ etc... $\quad x$-simple regions $f_1(y,z) \le f_2(y,z)$ cont surfaces

Ex: Re-write $\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx$ as $\iiint_R f(x,y,z)\,dx\,dy\,dz$

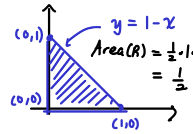
We get:
$0 \le x \le 1$
$0 \le y \le x$
$0 \le z \le y$

$\Rightarrow 0 \le z \le y \le x \le 1 \quad = \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz$

Allow each var as LARGE/SMALL as possible while satisfying old bounds

Ex: Find the average of $f(x,y) = e^{x+y}$ over Δ w/ vertices $(0,0)$, $(0,1)$, $(1,0)$

Find Area



$\text{Area}(R) = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$

Find the Integral

$\int_0^1 \int_0^{1-x} e^{x+y}\,dy\,dx = \int_0^1 \left[e^{x+y}\right]_0^{1-x} dx$

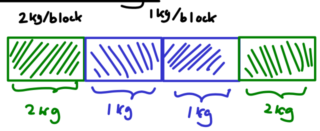
$= \int_0^1 e^{x+1-x} - e^{x+0}\,dx = \int_0^1 e - e^x\,dx$

$= [ex - e^x]_0^1 = (e-e) - (0 - e^0) = 1$

put it together

$\text{Avg} = \frac{1}{\text{Area}(R)} \iint_R e^{x+y}\,dA = \frac{1}{1/2}\cdot 1 = 2$

Mass and Density



How much does it weigh?

$2+1+1+2 = 6\text{kg}$

$\text{Density} = \dfrac{\text{weight}}{\text{area}} \Rightarrow \text{Mass} = \dfrac{2\text{kg}}{\text{block}}(2\text{ blocks}) + \dfrac{1\text{kg}}{\text{block}}(\text{block})$

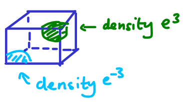
$= 2\cdot 2 + 2\cdot 1 = 6\text{kg}$

One Integration of:

$\iiint_R f(x,y,z)\,dV$ is the "weight of region R w/ density $f(x,y,z)$"

Volume is the "weight" obtained using density $f(x,y,z) = 1$ (uniform)

Ex: Calculate the mass of a cube w/ density $f(x,y,z) = e^{x+y+z}$ and co-ordinates $[-1,1]\times[-1,1]\times[-1,1]$



$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^{x+y+z}\,dx\,dy\,dz = \int_{-1}^{1}\int_{-1}^{1}\left[e^{x+y+z}\right]_{-1}^{1} dy\,dz$

$= \int_{-1}^{1}\int_{-1}^{1} e^{1+y+z} - e^{y+z-1}\,dy\,dz = \int_{-1}^{1}\left[e^{1+y+z} - e^{y+z-1}\right]_{-1}^{1} dz$

$= \int_{-1}^{1} \left(e^{1+1+z} - e^{1+z-1}\right) - \left(e^{1-1+z} - e^{-1+z-1}\right) dz$

$= \int_{-1}^{1} e^{2+z} - e^{z} - e^{z} - e^{z-2}\,dz = \int_{-1}^{1} e^{2+z} - 2e^{z} - e^{z-2}\,dz$

$= \int_{-1}^{1} e^{z}\left(e^{2} - 2 - e^{-2}\right) dz = \left[e^{z}\right]_{-1}^{1}\left(e^{2}-2-e^{-2}\right)$

$= \left(e - e^{-1}\right)\left(e^{2} - 2 - e^{-2}\right)$

The cube has the weight:

$\left(e - e^{-1}\right)\left(e^{2} - 2 - e^{-2}\right)$

Ex: Suppose a rod has density $\delta(x) = x^2 \,\frac{\text{kg}}{\text{m}}$ and runs from $x=-1$ to $x=1$

How much does it weigh?

We get

$\text{weight} = \int_{-1}^{1} \delta(x)\,dx$

$= \int_{-1}^{1} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = 2/3\text{ kg}$

Ex: Suppose a square metal plate has density $\delta(x,y) = x^2+y^2$ on $[-1,1]\times[-1,1]$. Find its mass

$\int_{-1}^{1}\int_{-1}^{1} \delta(x,y)\,dx\,dy$

$\text{kg} = \underbrace{\frac{\text{kg}}{\text{m}^2}}_{\delta}\ \underbrace{\text{m}}_{dx}\cdot\underbrace{\text{m}}_{dy}$

$= \int_{-1}^{1}\int_{-1}^{1} x^2+y^2\,dx\,dy$

$= \int_{-1}^{1}\left[\frac{1}{3}x^3 + y^2x\right]_{-1}^{1} dy = \int_{-1}^{1} \frac{2}{3} + 2y^2\,dy = \left[\frac{2}{3}y + \frac{2}{3}y^3\right]_{-1}^{1} = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

Summary

- Density
- Averages
- Triple Integrals
- Fubini
- Sketching in 3D
  - ↳ none on the exam, but 20 will be
- min-max inequality