Limits ^ Continuity

(1)

Recall the def^n

$$\lim_{\bar{x} \to \bar{y}} f(\bar{x}) = \bar{z} \iff \forall \varepsilon > 0, \exists \delta > 0 \mid \underbrace{\|\bar{x} - \bar{y}\| < \delta}_{\bar{x} \text{ close to } \bar{y}} \Rightarrow \underbrace{\|f(\bar{x}) - \bar{z}\| < \varepsilon}_{f(\bar{x}) \text{ close to } \bar{z}}$$

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(2)

Def^n: $f: \mathbb{R}^n \to \mathbb{R}^k$ is CONTINUOUS @ $\bar{x} = \bar{y}$ if

$$\lim_{\bar{x} \to \bar{y}} f(\bar{x}) = f(\bar{y})$$

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(3)

Def^n: $f(\bar{x})$ is CONTINUOUS on $D \subseteq \mathbb{R}^n$ (↳ domain) if

$$\lim_{\bar{x} \to \bar{d}} f(\bar{x}) = f(\bar{d}) \quad \forall \bar{d} \in D$$

we say "$f(\bar{x})$ is continuous" if $f(\bar{x})$ is continuous on it's domain

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(4)

Key Facts: If $f(\bar{x})$ and $g(\bar{x})$ are cnts @ $\bar{x} = \bar{y}$ then the following are also cnts @ $\bar{x} = \bar{y}$

1. $kf(\bar{x})$ for k constant
2. $f(\bar{x}) + g(\bar{x})$
3. $f(\bar{x}) \cdot g(\bar{x})$

Proof of above:

1. • If $k = 0$ then $kf(\bar{x}) = \bar{0} \ \forall \bar{x}$
   • suppose $k \neq 0$
   By hypothesis, $\lim_{\bar{x} \to \bar{y}} f(\bar{x}) = f(\bar{y})$

$$\forall \varepsilon_1 > 0, \exists \delta_1 > 0 \mid \|\bar{x} - \bar{y}\| < \delta_1 \Rightarrow \|f(\bar{x}) - f(\bar{y})\| < \varepsilon_1$$

WTS we need $\lim_{\bar{x} \to \bar{y}} kf(\bar{x}) = kf(\bar{y})$

Given $\varepsilon > 0$, pick $\delta = \delta_1$ for $\varepsilon_1 = \frac{\varepsilon}{|k|} > 0$

$$\|\bar{x} - \bar{y}\| < \delta = \delta_1 \Rightarrow \|f(\bar{x}) - f(\bar{y})\| < \varepsilon_1$$
$$\Rightarrow \|f(\bar{x}) - f(\bar{y})\| < \frac{\varepsilon}{|k|}$$
$$\Rightarrow |k| \, \|f(\bar{x}) - f(\bar{y})\| < |k| \left(\frac{\varepsilon}{|k|}\right)$$
$$\Rightarrow \|kf(\bar{x}) - kf(\bar{y})\| < \varepsilon$$

> why $|k|$ and not k?
> • $|k|$ ensures we can multiply by k ^ inequality remains preserved

2. By hypothesis

$$\lim_{\bar{x} \to \bar{y}} f(\bar{x}) = f(\bar{y}) \text{ and } \lim_{\bar{x} \to \bar{y}} g(\bar{x}) = g(\bar{y})$$

we get: $\forall \varepsilon_f > 0, \exists \delta_f > 0 \mid \|\bar{x} - \bar{y}\| < \quad \Rightarrow \|f(\bar{x}) - f(\bar{y})\| <$

$\forall \varepsilon_g > 0, \exists \delta_g > 0 \mid \|\bar{x} - \bar{y}\| < \quad \Rightarrow \|g(\bar{x}) - g(\bar{y})\| < \delta_g$