MAT B41 — Week 1a

Welcome!

→ Syllabus

→ Pre-course Survey.

Our course webpage: http://pgadey.ca

Vectors

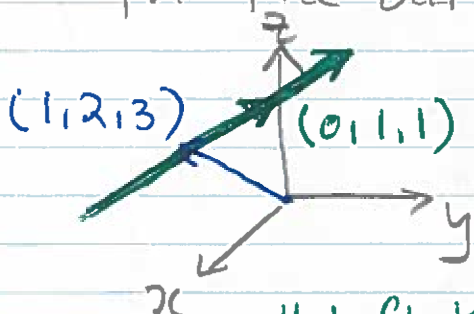
In this course

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$$

? The standard basis vectors $\vec{i}\ \vec{j}\ \vec{k}$ come from physics.

Ex: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\vec{j} = \begin{bmatrix} 1 \\ 2+3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$

Ex: Write the parametric equation of the line passing through $(1,2,3)$ in the direction $\vec{j}+\vec{k}$.



$$(x,y,z) = (1,2,3) + t(0,1,1)$$
$$= (1, 2+t, 3+t).$$

Left thumb is x-axis

Lengths and Angles

The Pythagorean Theorem says:

$$\|(x,y,z)\|^2 = x^2+y^2+z^2$$

Nota: The notation $\|\vec{v}\|$ means the LENGTH of the vector $\vec{v}$.

? $\vec{v}$ has a harpoon

Ex: Find the length $\|\vec{i}+2\vec{j}\|$

$$\|\vec{i}+2\vec{j}\| = \sqrt{1^2+2^2} = \sqrt{5}$$

Ex: Find the length $\left\|\begin{bmatrix}3\\4\end{bmatrix}\right\|$

$$= \sqrt{3^2+4^2}$$
$$= \sqrt{25} = 5.$$

Defn: The DOT PRODUCT $\vec{u}\cdot\vec{v}$ is

$$\begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}a\\b\\c\end{bmatrix} = ax+by+cz$$

Fact: $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

Ex: Find the angle between $\vec{u}=\begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}1\\0\end{bmatrix}$.

$$\vec{u}\cdot\vec{v} = 1 = \sqrt{2}\cdot 1\cos\theta$$
$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \pi/4.$$

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Def^n: Two vectors are ORTHOGONAL if the form an angle of $\theta = \pi/2$

! $\theta = \pi/2 \Rightarrow \vec{u}\cdot\vec{v} = 0$

Two vectors are PARALLEL if: $\vec{u} = \lambda\vec{v}$ for some number λ.

Ex: Find two non-parallel vectors that are orthogonal to $(1,2,3)$.

orthogonal $\Rightarrow \vec{u}\cdot(1,2,3) = 0$

Let $\vec{u} = (x,y,z)$ we get:

$$x + 2y + 3z = 0$$

We may pick: $(x,y,z) = (3,0,-1)$

$(x,y,z) = (0,3,-2)$

These are non-parallel since:

$$(0,3,-2) \neq \lambda(0,3,-2)$$

Ex: Find all values x so that:

$\vec{u}(x) = (5,x,3)$ and

$\vec{v}(x) = (x,x,2)$

are orthogonal.

We need $\vec{u}(x)\cdot\vec{v}(x) = 0$

$$\vec{u}(x)\cdot\vec{v}(x) = (5,x,3)\cdot(x,x,2)$$

$$= 5x + x^2 + 6 = (x+2)(x+3)$$

Thus, $x = -2$ OR $x = -3$.

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Lines and Planes

Recall the parametric equation of a line:

$$\boxed{\vec{x} = \vec{p} + t\vec{d}}$$

$\vec{p}$ = initial point

$\vec{d}$ = direction

t = parameter

Ex: Find the line passing through: $(1\ 2\ 3)$ and $(4\ 5\ 6)$

Find $\vec{p}$ and $\vec{d}$.

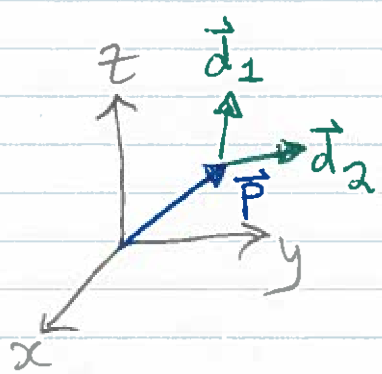
Pick $\vec{d} = (4\ 5\ 6) - (1\ 2\ 3) = (3\ 3\ 3)$

$\vec{p} = (1\ 2\ 3)$

We obtain: $\vec{x} = (1\,2\,3) + t(3\ 3\ 3)$

Defn: The PARAMETRIC EQUATION of a PLANE

$$\vec{x} = \vec{p} + s\vec{d}_1 + t\vec{d}_2$$



Go to $\vec{p}$ and then add stretched copies of $\vec{d}_1$ and $\vec{d}_2$

Ex: Give a parametric equation for the plane containing $(1\ 0\ 0)$, $(0\ 1\ 0)$, and $(0\ 0\ 1)$

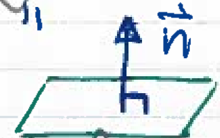
Pick $\vec{p}$, $\vec{d}_1$, and $\vec{d}_2$

$\vec{p} = (1\ 0\ 0)$ $\qquad \vec{d}_1 = (0\,1\,0) - (1\,0\,0) = (-1\ 1\ 0)$

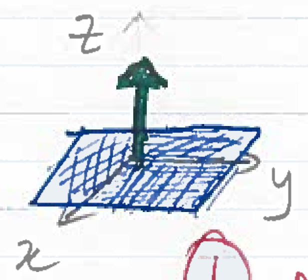
$\vec{d}_2 = (0\,0\,1) - (1\,0\,0) = (-1\ 0\ 1)$

The Normal Form of Planes

We can imagine a plane as the set of vectors orthogonal to a given vector $\vec{n}$. "the normal"



Ex: Write the xy-plane in normal form.



Every vector in the xy-plane is orthogonal to the z-axis.

$$xy\text{-plane} = \{\vec{v} : \vec{v}\cdot(0\ 0\ 1) = 0\}$$

(!) A plane in this form will always contain $\vec{0}$ since $\vec{v}\cdot\vec{0} = 0$.

Defn: The NORMAL FORM of a PLANE CONTAINING $\vec{p}$ with NORMAL $\vec{n}$

"π for plane" $\to$ $$\pi = \{\vec{v} : (\vec{v}-\vec{p})\cdot\vec{n} = 0\}$$

Ex: Express the plane through $(1,2,3)$ and parallel to the yz-plane in normal form.

Find the normal. x-axis perpendicular to the yz-plane

$$\vec{n} = (1\ 0\ 0)$$

Apply the formula

$$\pi = \{\vec{v} : (\vec{v}-(1\ 2\ 3))\cdot(1\ 0\ 0) = 0\}$$

Equivalently: $((x\ y\ z)-(1\ 2\ 3))\cdot(1\ 0\ 0) = 0$

$$\Leftrightarrow x - 1 = 0$$

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Ex: Determine where the lines:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \text{ intersect.}$$

Set the equations equal.

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} 2+3t \\ -1+t \\ 1+6t \end{bmatrix} = \begin{bmatrix} -1+3s \\ -2+s \\ 0+s \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} 3t-3s \\ t-s \\ 6t-s \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix}$$

Solve the system

$$\begin{array}{ccc} t & s & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 3 & -3 & -3 \\ 1 & -1 & -1 \\ 6 & -1 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 6 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 - 6R_1} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + R_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

Thus, $t=0$ and $s=1$ works.

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* Inner product is dot product

Ex: Determine where the line

$$\ell: \vec{x} = (2\ 1\ 0) + t(1\ 1\ 0)$$

meets the plane:

$$\pi: x + 2y + 3z = 0.$$

\# Write the coordinates of a point on ℓ

$$\vec{x} = (2+t \quad 1+t \quad 0)$$

\# Input ℓ in to π

$$(2+t) + 2(1+t) + 3 \cdot 0 = 0$$

$$\Leftrightarrow 4 + 3t = 0 \Leftrightarrow t = -\frac{4}{3}$$

Thus, $\vec{x} = (2\ 1\ 0) + \left(-\frac{4}{3}\right)(1\ 1\ 0)$

$$= \left(\tfrac{2}{3} \quad -\tfrac{1}{3} \quad 0\right)$$

Ex: Find a normal to the plane containing $(1\ 1\ 1)$ $(2\ 1\ 1)$ $(2\ 3\ 0)$.

\# Find vectors in the plane

$$\vec{u} = (2\ 1\ 1) - (1\ 1\ 1) = (1\ 0\ 0)$$

$$\vec{v} = (2\ 3\ 0) - (1\ 1\ 1) = (1\ 2\ -1)$$

\# Find $\vec{w}$ orthogonal to $\vec{u}$ and $\vec{v}$.

Suppose $\vec{w} = (x, y, z)$.

We need

$$0 = \vec{u} \cdot \vec{w} = x$$

$$0 = \vec{v} \cdot \vec{w} = x + 2y - z$$

We pick [illegible]