Complex Numbers

? Complex means "having multiple parts".

Def^n: The COMPLEX NUMBERS are:

$$\mathbb{C} = \{a+bi : a,b \in \mathbb{R}\}$$

and $i^2 = -1$

Def^n: $(a+bi)+(c+di) = (a+c)+(b+d\,i)$

$$(a+bi)(c+di) = ac + adi + bci + bdi^2 \quad \# i^2=-1$$
$$= (ac-bd) + (ad+bc)i$$

! Don't memorize this formula.

Ex: Express $(1+2i)(3+4i)$ in the form $a+bi$.

$$(1+2i)(3+4i)$$
$$= 1\cdot3 + 1\cdot 4i + 3\cdot 2i + 2\cdot 4\cdot i^2 \quad \# i^2=-1$$
$$= (3-8)+(4+6)i = -5+10i$$

Ex: Calculate $(1+i)-(2+i)^2$

$$(1+i)-(2+i)^2$$
$$= (1+i) - (2^2 + 2\cdot 2\cdot i + i^2) \quad \# i^2 = -1$$
$$= 1+i-4-4i+1$$
$$= -2-3i$$

Question: What is the intuitive explanation for complex multiplication?

Defn: The REAL PART of $a+bi$ is $\mathrm{Re}(a+bi)=a$
IMAGINARY PART $\mathrm{Im}(a+bi)=b$

Defn: The CONJUGATE of $a+bi$ is $\overline{a+bi}=a-bi$

Thm: If z is a complex number then $z\bar{z}$ is a real number we denote $\|z\|^2$.

Pf: Let $z=a+bi$. We get

$$\begin{aligned}\|z\|^2 = z\bar{z} &= (a+bi)(a-bi)\\ &= a^2 - abi + abi - b^2 i^2\\ &= a^2+b^2\end{aligned}$$

Nota: If x is a real number we write $x = x+0i$ for the complex number x.

Thm: If $z\neq 0$ then there is w so that $zw=1$.

Pf: Take $w=\frac{1}{\|z\|^2}\cdot\bar{z}$. We get:

$$zw = \frac{1}{\|z\|^2}\cdot z\bar{z} = \frac{1}{\|z\|^2}\cdot\|z\|^2 = 1.$$

Ex: Find w so that $(1+i)z=1$.

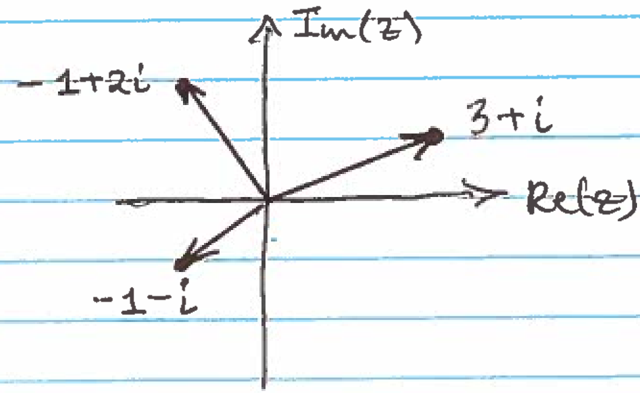
$$w = \frac{1}{1^2+1^2}\cdot(1-i) = \frac{1}{2}-\frac{1}{2}i$$

check $zw=1$

$$(1+i)\left(\tfrac{1}{2}-\tfrac{1}{2}i\right) = \tfrac{1}{2}-\tfrac{1}{2}i+\tfrac{1}{2}i-\tfrac{1}{2}i^2 = \tfrac{1}{2}+\tfrac{1}{2} = 1.$$

The Geometry of the Complex Numbers

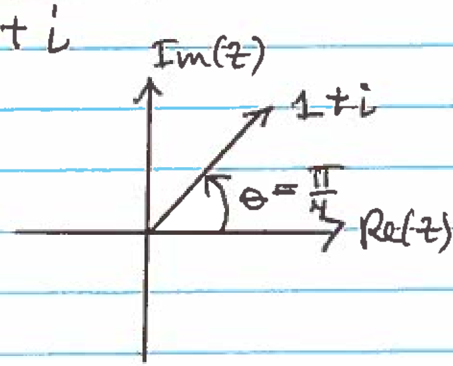
Defn: The COMPLEX PLANE (ARGAND DIAGRAM) is the usual Cartesian plane where the y-axis measures Im(z) and the x-axis measures Re(z)



Defn: The MODULUS of a complex number z is its length in the Argand diagram.

The ARGUMENT of z is the angle it forms with the Re(z) axis measured counter-clock-wise.

Ex: Calculate the modulus and argument of $z = 1+i$



$\|z\|^2 = 1^2 + 1^2$

$\Rightarrow \|z\| = \sqrt{2}$

$\theta = \arctan\left(\frac{1}{1}\right)$

$= \arctan(1)$

$= \frac{\pi}{4}.$

Euler's Formula

Recall, $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \dots -$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \dots$$

To prove these, apply Taylor's Formula:

$$\text{"} \; f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \; \text{"}$$

Thm: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Pf:

$$e^{i\theta} = 1 + \frac{(i\theta)^1}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \dots$$

$$= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \dots\right]$$

Seperate ~~complex~~ real and imaginary terms

$$+ i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \dots\right]$$

$$= \cos\theta + i\sin\theta$$

Trig Identities à la Euler

Thm: $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$
$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

Pf: $\cos(\alpha+\beta) + i\sin(\alpha+\beta)$

$= e^{i(\alpha+\beta)} = e^{i\alpha + i\beta}$ # Euler's formula

$= e^{i\alpha}e^{i\beta}$

$= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta))$

$= \cos(\alpha)\cos(\beta) + i\sin(\beta)\cos(\alpha) + i\sin(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ # $i^2 = -1$

$= [\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + i[\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)]$

(!) We get both identities at once by equating real and imaginary parts.

Thus, $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ and
$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$

Exercise: Use $e^{i(3\theta)} = (e^{i\theta})^3$ to find the formulae for $\sin(3\theta)$ and $\cos(3\theta)$.

Polynomials

Def^n: A POLYNOMIAL WITH REAL COEFFICIENTS is an expression

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where n is a natural number, the a_i are real numbers, and $a_n \neq 0$. n is the DEGREE and the numbers a_i are COEFFICIENTS

Def^n: A ROOT of $p(x)$ is a number z so that:
$$p(z) = 0$$

Ex: Find the roots of $p(x) = x^2 - 5x + 4$.

$p(x) = (x-1)(x-4) \Rightarrow x = 1$ ad $x = 4$ are the roots of $p(x)$

Ex: Find the roots of $q(x) = x^2 + 1$.

$q(x) = x^2 - i^2 = (x-i)(x+i)$
$\Rightarrow x = i$ and $x = -i$ are the roots of $q(x)$.

NB: $q(x)$ has no real roots since:
$$x^2 \geq 0 \Rightarrow x^2 + 1 > 0.$$

Thus $q(x)$ only has complex roots.

The Quadratic Formula

Thm: If $p(x) = ax^2 + bx + c$ is a quadratic polynomial then $p(x)$ has roots:

$$x \in \left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\}$$

Pf:

Suppose $ax^2 + bx + c = 0$

By hypothesis, $a \neq 0$ ad we may divide by it.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

$$\Rightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, $x \in \left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\}$

Thm (The Fundamental Theorem of Algebra)

Every non-constant polynomial with complex coefficients has a complex root.

Corollary: Every polynomial of degree n has n roots (counted with multiplicity)

Ex: The polynomial $p(x) = (x-1)^1(x-2)^2(x-3)^3$ has degree six and roots $x = 1, 2, 3$.

However, the root $x=3$ has multiplicity three
$x=2$ two
$x=1$ one

Thus, $p(x)$ has six roots with multiplicity

Defn: The MULTIPLICITY of a root $(x-\alpha)$ is the largest power k such that: $(x-\alpha)^k \mid p(x)$.

Defn: We say $p(x) \mid q(x)$ if:

$$q(x) = p(x) \cdot k(x)$$

for some polynomial $k(x)$.

Ex: $(x-1) \mid (x^2 - 3x + 2)$ since

$$(x^2 - 3x + 2) = (x-1)(x-2)$$

"Sketch" of Proof of Fund Thm:

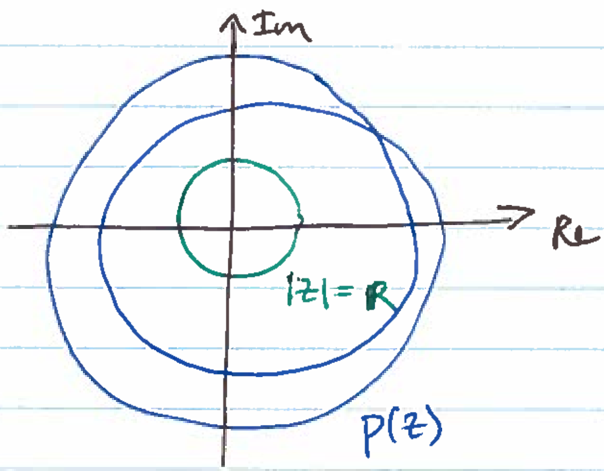
Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$

be a polynomial with complex coefficients.

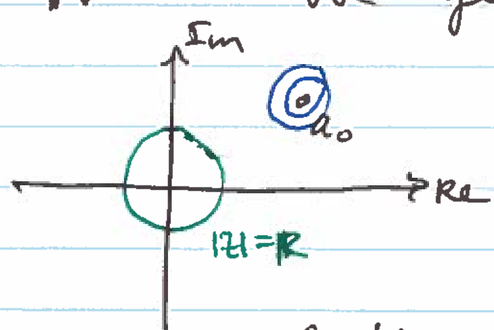
For $|z| = R$ very large we have that

$\{ p(z) : |z| = R \}$ is a loop of

radius $\approx |a_n| \cdot |R|^n$



As we deform $R \to 0$ we get



The loop must pass through the origin to do this.

Polynomial Long Division

Recall long division from highschool,

Ex: Divide 341 by 7.

$$
\begin{array}{r|l}
 & \;\;48 \\
\hline
7 & 341 \\
 & -28\downarrow \\
\hline
 & \;\;61 \\
 & -56 \\
\hline
 & \;\;\;\;5
\end{array}
$$

Thus,
$341 = 7 \cdot 48 + 5$

Ex: Divide $x^3 - 10x^2 + 31x - 30$ by $x - 2$.

$$
\begin{array}{r|l}
 & x^2 - 8x + 15 \\
\hline
x-2 & x^3 - 10x^2 + 31x - 30 \\
 & \ominus\, x^3 - 2x^2 \\
\hline
 & \quad -8x^2 + 31x \\
 & \ominus\, -8x^2 + 16x \\
\hline
 & \qquad 15x - 30 \\
 & \ominus\, 15x - 30 \\
\hline
 & \qquad\quad 0
\end{array}
$$

Ex: Divide $x^2 - 8x + 15$ by $x - 3$.

$$
\begin{array}{r|l}
 & x - 5 \\
\hline
x-3 & x^2 - 8x + 15 \\
 & \ominus\, x^2 - 3x \\
\hline
 & \quad -5x + 15 \\
 & \ominus\, -5x + 15 \\
\hline
 & \qquad 0
\end{array}
$$

Thus, $x^3 - 10x^2 + 31x - 30 = (x-2)(x-3)(x-5)$

<u>Thm</u>: If r is a complex number and $p(z)$ is a non-constant polynomial with complex coefficients, then there exists a polynomial $q(z)$ and a constant c such that

$$p(z) = (z-r)q(z) + c.$$

<u>Pf</u>: We proceed by induction on degree.

If $n=1$ then $p(z) = az + b$.

We get
$$\begin{aligned} p(z) &= (z-r)\cdot a + (ar+b) \\ &= (z-r)q(z) + c. \end{aligned}$$

Suppose the claim holds for all degrees $\leq n$.

$$\begin{aligned} p(z) &= a_{n+1}z^{n+1} + a_n z^n + \dots + a_0 \\ &= (z-r)(a_{n+1}z^n) + \underbrace{r a_{n+1} z^n + a_n z^n + \dots + a_0}_{\deg \leq n} \end{aligned}$$

apply the inductive hypothesis to the degree $\leq n$ terms.

$$\begin{aligned} &= (z-r)(a_{n+1}z^n) + (z-r)q_n(z) + c \\ &= (z-r)(a_{n+1}z^n + q_n(z)) + c \\ &= (z-r)q_{n+1}(z) + c. \end{aligned}$$

Thm: r is a root of $p(z)$
$\iff (z-r) \mid p(z)$

Pf: Suppose, $p(r) = 0$.

We have that $p(z) = (z-r)\,q(z) + c$
It follows,

$$0 = p(r) = \underbrace{(r-r)\,q(r) + c}_{= c}$$

Thus $c = 0$ ad

$$p(z) = (z-r)\,q(z).$$

Suppose, $(z-r) \mid p(z)$.

We have $p(z) = (z-r)\,q(z)$

Thus, $p(r) = (r-r)\,q(r) = 0$.

Thm (De Moivre)

$$[r(\cos\theta + i\sin\theta)]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

Pf:

$$[r(\cos\theta + i\sin\theta)]^n$$
$$= r^n[\cos\theta + i\sin\theta]^n$$
$$= r^n[e^{i\theta}]^n$$
$$= r^n(e^{in\theta}) = r^n e^{i(n\theta)}$$
$$= r^n[\cos(n\theta) + i\sin(n\theta)]$$

Ex: Calculate $(1+i)^{100}$

$$(1+i)^{100} = \left(\sqrt{2}\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right)\right)^{100}$$
$$= 2^{50}\left(\cos\left(\tfrac{\pi}{4}\cdot 100\right) + i\sin\left(\tfrac{\pi}{4}\cdot 100\right)\right)$$
$$= 2^{50}\left(\cos(25\pi) + i\sin(25\pi)\right)$$
$$= 2^{50}(-1 + i\cdot 0) = -2^{50}$$

Roots of Unity

Def'n: The n^{th} ROOTS OF UNITY are the solutions of the polynomial

$$p_n(z) = z^n - 1.$$

Ex: The second roots of unity are:

$$P_2(z) = z^2 - 1 = (z-1)(z+1)$$
$$\Rightarrow z = -1, 1.$$

Ex: The third roots of unity are:

$$P_3(z) = z^3 - 1 = (z-1)(z^2+z+1)$$
$$\Rightarrow z = 1, \frac{-1-\sqrt{-3}}{2}, \frac{-1+\sqrt{-3}}{2}$$

Thm: The n^{th} roots of unity may be calculated by solving:

$$\begin{cases} \cos(n\theta) = 1 \\ \sin(n\theta) = 0 \end{cases}$$

Thus,

$$z = 1, \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right),$$
$$\cos\left(2\cdot\frac{2\pi}{n}\right) + i\sin\left(2\frac{2\pi}{n}\right),$$
$$\vdots$$
$$\cos\left((n-1)\cdot\frac{2\pi}{n}\right) + i\sin\left((n-1)\frac{2\pi}{n}\right).$$