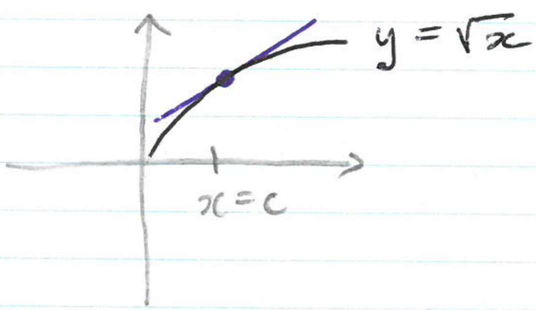
MAT 133 — Week 13c

Approximation (§5.10)

We develop a general technique for approximating "nice" functions.

Idea: The tangent line at $x=c$ approximates $f(x)$ near $x=c$.



Ex: Find the tangent line to $y=\sqrt{x}$ at $x=4$.

\# Find the slope

$$m=f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{4}$$

\# Find the y-intercept

$$y=\frac{1}{4}x+b \Rightarrow 2=\frac{1}{4}\cdot 4+b$$

$$b=1$$

Thus, $y=\frac{1}{4}x+1$.

Ex: Use the tangent line to approximate $\sqrt{3.9}$ and $\sqrt{4.1}$ and compare with the real values.

$$\sqrt{3.9}\approx\frac{1}{4}(3.9)+1$$
$$=1.975$$
$$\approx 1.9748$$

$$\sqrt{4.1}\approx\frac{1}{4}(4.1)+1$$
$$=2.025$$
$$\approx 2.0248$$

Ex: Use linear approximation at $x=1$ to approximate

$$\ln(0.9) \text{ and } \ln(e^{10})$$

Find the tangent line at $x=1$.

$$m = f'(1) = \frac{1}{1} = 1$$

$$y = 1 \cdot x + b \implies \ln(1) = 1 \cdot 1 + b$$
$$\implies 0 = 1 + b$$
$$\implies y = x - 1.$$

Approximate $\ln(0.9)$

$$\ln(0.9) \approx 0.9 - 1 = -0.1$$
$$\approx -0.10536\ldots$$

Approximate $\ln(e^{10})$

$$\ln(e^{10}) \approx e^{10} - 1$$

! This is huge.
$\ln(e^{10}) = 10$
$e^{10} - 1 \approx 22025.44$

We note that the approximation becomes inaccurate as we move farther away from $x=1$.

Our approximation is "good" near $x=1$.

Ex: Find the linear approximation at $x=10$ of $y = 5x+6$.

We get that the line approximates itself.

MAT 133 — Week 13c

Σ-notation Refresher

$$f(5)+f(6)+f(7) = \sum_{k=5}^{7} f(k)$$

UPPER BOUND: 7 · FUNCTION: $f(k)$ · VARIABLE: k · LOWER BOUND: 5

Σ-notation writes sums unambiguously.

Ex: Write $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + 99\cdot 100$ using Σ-notation

$$\sum_{k=1}^{99} k(k+1) = 1\cdot 2 + 2\cdot 3 + \dots + 99\cdot 100.$$

Ex: Write out $\sum_{k=5}^{10} 2^k$ but do not evaluate it.

$$\sum_{k=5}^{10} 2^k = 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10}$$

Nota: The FACTORIAL of n is:

$$n! = n(n-1)(n-2)\dots 3\cdot 2\cdot 1$$

Ex: Calculate $4! = 4\cdot 3\cdot 2\cdot 1 = 24.$

Ex: Find the n^{th} derivative of $f(x) = x^n$

$$f^{(n)}(x) = n\cdot(n-1)(n-2)\cdots 2\cdot 1\cdot x^0$$

$$= n!$$

Taylor Polynomials

Defⁿ: The n^{th} ORDER APPROXIMATION at $x=c$

$$P_{n,c}(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Idea: $P_{n,c}(x)$ has the same n derivatives as f at c.

Ex: Find the 2^{nd} order approximation at $x=0$ of $f(x) = 1+2x+x^2$

$$P_{2,0}(x) = f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2$$

$$= 1 + \frac{2}{1!}x + \frac{2}{2!}x^2 = 1+2x+x^2.$$

Idea: The n^{th} order approximation of a poly-nomial of degree n is itself.

Ex: Find the 4^{th} order approximation at $x=0$ of $f(x) = e^x$

$$P_{4,0}(x) = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$