A31 Wk 8a

This week: § 2.3 – 2.4 (Derivatives!)

We continue to develop rules for working with derivatives of functions.

Thm (Scaling): If $g(x) = k \cdot f(x)$ then $g'(x) = k \cdot f'(x)$

Pf: $$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{k \cdot f(x+h) - k \cdot f(x)}{h}$$

$$= \lim_{h \to 0} k \cdot \left[\frac{f(x+h) - f(x)}{h} \right]$$ # limit scaling

$$= k \cdot f'(x)$$

Thm (Addition): If $F(x) = f(x) + g(x)$ then $F'(x) = f'(x) + g'(x)$.

Pf: $$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right]$$

$$= f'(x) + g'(x).$$

Scaling + Addition ⇒ Difference

Also, "linearity"

We give another proof of the power rule for whole number powers.

<u>Thm</u>: If $k \in \mathbb{N}$ and $f(x) = x^k$ then $f'(x) = k \cdot x^{k-1}$.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^k - x^k}{(x+h) - x}$$

<u>Discuss</u>: convince your neighbour that this is true!

$$= \lim_{y \to x} \frac{y^k - x^k}{y - x}$$

<u>NB</u>: k terms

$$= \lim_{y \to x} \frac{(y-k)(\overbrace{y^{k-1} + xy^{k-2} + \cdots + x^{k-1}}^{k \text{ terms}})}{(y-k)}$$

$$= \lim_{y \to x} y^{k-1} + xy^{k-2} + x^2y^{k-3} + \cdots + x^{k-1}$$

$$= x^{k-1} + x \cdot x^{k-2} + x^2 x^{k-3} + \cdots + x^{k-1}$$

$$= k\,x^{k-1}$$

This proof is just as valid and important as the proof we gave in class last week.

<u>Note</u>: This proof only works for $k \in \mathbb{N}$ since that is where we can use the factorization formula.

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Thm: If $F(x) = \frac{f(x)}{g(x)}$ then $F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Pf: $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

common denominator $= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)\cdot g(x+h)}{h\, g(x+h)\, g(x)}$

introduce cross-terms $= \lim_{h\to 0} \frac{f(x+h)g(x) + g(x)f(x) - g(x)f(x) - f(x)g(x+h)}{h\, g(x+h)\, g(x)}$

$$= \lim_{h\to 0} \frac{g(x)[f(x+h) - f(x)]}{h\, g(x+h)\cdot g(x)} + \frac{f(x)[g(x) - g(x+h)]}{h\, g(x+h)\, g(x)}$$

NB: sign change $= \lim_{h\to 0} \frac{g(x)}{g(x+h)\, g(x)} \cdot \left[\frac{f(x+h) - f(x)}{h}\right] - \frac{f(x)}{g(x+h)g(x)}\left[\frac{g(x+h) - g(x)}{h}\right]$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Note the formal similarity to the proof of the product rule.

① Set up derivative
② Introduce cross terms
③ Factor appropriately.

Summary: Things we have proven about derivatives.

- slope = slope of tangent = limit of slopes of secant lines
- $\frac{d}{dx}(mx+b) = m.$
- $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
- $k \in \mathbb{N} \implies \frac{d}{dx}(x^k) = k \cdot x^{k-1}.$
- $\frac{d}{dx}(k \cdot f(x)) = k \frac{d}{dx} f(x)$
- $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$
- * $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Derivatives and Composition (§2.4)

We need the derivative $\frac{d}{dx}[f(g(x))]$.

Ex: Suppose you make widgets for free.
You can make 2 widgets/hr
and you can sell them for
3 dollars/widget.

How much can you earn per hour?

$(2 \text{ widgets/hr})(3 \$/\text{widget})$

$= 6 \$/\text{hr}.$ Discuss: Why?

"The Math"

You have $2t$ widgets at
time t. You can sell
n widgets for $3n$ dollars.

Thus,
you make $3(2t)$ dollars.
(at time t).

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Thm: (Chain Rule) p202

If $F(x) = f(g(x))$

then $$\frac{d}{dx}\left[F(x)\right] = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$$

NB: The "units" $g(x)$ cancel out.
The "rates" cancel.

Pf:

$$\frac{d}{dx}F(x) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{(x+h) - x}$$

$g(y) = g(x+h)$

$$= \lim_{y\to x} \frac{f(g(y)) - f(g(x))}{y - x}$$

$$= \lim_{y\to x} \frac{f(g(y)) - f(g(x))}{g(y) - g(x)} \cdot \frac{g(y) - g(x)}{y - x}$$

Differentiability of $g(x)$

$$= \lim_{y\to x}\left[\frac{f(g(y)) - f(g(x))}{g(y) - g(x)}\right] \lim_{y\to x}\left[\frac{g(y) - g(x)}{y - x}\right]$$

$$= f'(g(x)) \cdot g'(x)$$

Ex: $F(x) = f(g(x))$ for $f(x) = \sqrt{x}$
$g(x) = x^2+1$

write out the function explicity

$$F(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x^2+1}$$

Take the derivative explicity

$$F'(x) = f'(g(x))\, g'(x) \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{g(x)}} \qquad g'(x) = 2x$$

$$= \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

"Pf":

$$\lim_{h \to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{(x+h) - x}$$

conjugate.

$$= \lim_{h \to 0} \frac{[(x+h)^2+1] - [x^2+1]}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

expand

$$= \frac{\cancel{x^2} + 2xh + h^2 + \cancel{1} - \cancel{x^2} - \cancel{1}}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

Ex: Compute the derivate of $\cos(x^2+2x+1)$

$$\frac{d}{dx}\left[\sin(x^2+2x+1)\right] = \frac{d\sin(x^2+2x+1)}{d(x^2+2x+1)}\,\frac{d(x^2+2x+1)}{dx}$$

$$= \cos(x^2+2x+1)\left[2x+2\right]$$

Generalization:

$$\frac{d\,f(g(h(x)))}{dx} = \frac{d\,f(g(h(x)))}{d\,g(h(x))}\cdot\frac{d\,g(h(x))}{d\,h(x)}\cdot\frac{dh(x)}{dx}$$

$$= f'(g(h(x)))\cdot g'(h(x))\cdot h'(x).$$

The chain rule applies to all lengths of compositions

Ex: $f(x) = ((x+1)^2+5)^3$

$$f'(x) = 3((x+1)^2+5)^2\cdot\frac{d}{dx}\left[(x+1)^2+5\right]$$

$$= 3((x+1)^2+5)^2\left[2(x+1)\right]$$

NB: You need to practice chain rule calculations like this.

Two Detours:

Higher-Order Derivatives:

$f'(x)$ = derivative of $f(x)$

$f''(x)$ = derivative of $f'(x)$

$f'''(x)$ = derivative of $f''(x)$.

$$f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x)$$

$$y = f(x) \iff \frac{d^{n+1}y}{dx^{n+1}} = \frac{d}{dx}\left[\frac{d^n y}{dx^n}\right]$$

Discuss: Compute all the derivatives of $y = x^4 + x^3 + x^2 + x + 1$.

What patterns do you observe?

Anti-Derivatives

Def^n: If $F'(x) = f(x)$ then $F(x)$ is an ANTI-DERIVATIVE of $f(x)$

Discuss: Find a general formula for finding anti-derivatives of polynomials.