MAT A31 Wk 6a

"Welcome back from Reading week!"
- travelling
- family
- good food
- studying

Two Key Facts about Limits

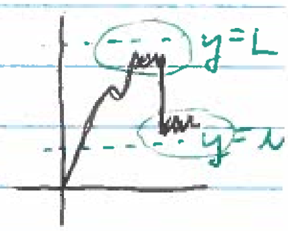
Thm: If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$

then $L = M$. (Uniqueness of limits.)

(*Discuss with your neighbour*)

Remember that "$\lim_{x\to c} f(x) = L$" is actually a proposition or theorem that needs proof!

Main Idea: This picture cannot happen: Your function would fail the vertical line test



Pf: Suppose ① $\lim_{x\to c} f(x) = L$ and ② $\lim_{x \to c} f(x) = M$.

If $L \neq M$ then $|L-M| > 0$. # There is a gap!

Apply ① to $\varepsilon = \frac{|L-M|}{10} > 0$ # $(10 > 2)$ # 10 is arbitrary

and get:

$\delta_1 > 0$ such that $|x-c| < \delta_1 \Rightarrow |f(x) - L| < \frac{|L-M|}{10}$

Apply ② to get:

$\delta_2 > 0$ such that $|x-c| < \delta_2 \Rightarrow |f(x) - M| < \frac{|L-M|}{10}$.

MAT A31 Wk 6a

Pf of Uniqueness of Limits: (con't)

Now pick $\delta < \min\{\delta_1 \text{ and } \delta_2\}$

We get: $0 < |x - c| < \delta \overset{①}{\Longrightarrow} |f(x) - L| < \frac{|L-M|}{10}$

$0 < |x - c| < \delta \overset{②}{\Longrightarrow} |f(x) - M| < \frac{|L-M|}{10}$.

We then have:

$$|L - M| = |L - M + f(x) - f(x)|$$

$$= |(L - f(x)) - (M - f(x))|$$

\# Triangle inequality $\leq |L - f(x)| + |M - f(x)|$

$$< \frac{|L-M|}{10} + \frac{|L-M|}{10} = \frac{2}{10}|L-M|$$

Thus, $|L - M| < \frac{2}{10}|L - M|$.

This contradicts the rules of arithmetic.

The Key Technical Steps

- Estimate $|f(x) - L|$ using $\frac{|L-M|}{10}$.

  $|f(x) - M|$ using $\frac{|L-M|}{10}$

- Triangle inequality.

MAT A31 Wk 6a

<u>Thm</u> (Well-Definedness of Limits)

$$\left(\lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L\right) \text{ if and only if } \left(\lim_{x\to c} f(x) = L\right)$$

* Discuss with your neighbour *
why do we prove this? Examples?

<u>Pf</u>: $[\Rightarrow]$ Suppose ① $\lim_{x\to c^-} f(x) = L$ and
only if

② $\lim_{x\to c^+} f(x) = L$

Apply ① to $\varepsilon > 0$ to get $\delta_1 > 0$:

$x \in (c-\delta_1, c) \Rightarrow |f(x) - L| < \varepsilon$.

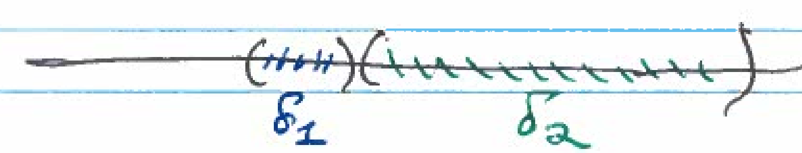
Apply ② to $\varepsilon > 0$ to get $\delta_2 > 0$:

$x \in (c, c+\delta_2) \Rightarrow |f(x) - L| < \varepsilon$

For $\varepsilon > 0$ take $\delta < \min\{\delta_1, \delta_2\}$

We get:

$0 < |x - c| < \delta \Rightarrow x \in (c-\delta, c) \Rightarrow |f(x) - L| < \varepsilon$
OR $x \in (c, c+\delta)$



Pf of Well-Definedness of Limits: (con't)

[$\Leftarrow$] Suppose $\lim_{x\to c} f(x) = L$.
if

For any $\varepsilon > 0$ there is $\delta > 0$:

$$0 < |x-c| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

Thus $x \in (c-\delta, c) \Longrightarrow |f(x) - L| < \varepsilon$
and $x \in (c, c+\delta) \Longrightarrow |f(x) - L| < \varepsilon$.

Therefore $\lim_{x\to c^-} f(x) = L$ and

$\lim_{x\to c^+} f(x) = L$.

Main ideas: [$\Longrightarrow$] Use 1 to get $\delta_1 > 0$
2 to get $\delta_2 > 0$.
Take $\delta < \min\{\delta_1, \delta_2\}$

[$\Leftarrow$] A punctured interval of radius $\delta > 0$ at $x = c$ is

$$\{x : 0 < |x-c| < \delta\}$$
$$= (c-\delta, c) \cup (c, c+\delta)$$

Limit Lemmas / continuity

Def^n: A function $f(x)$ is continuous at $x=c$ if:

$$\lim_{x\to c} f(x) = f(c)$$

A function is continuous if it is continuous at each point of its domain.

The Limit Lemmas simplify proofs of continuity.

Recall,

$$\begin{cases} \lim_{x\to c} f(x) = F \quad \lim_{x\to c} g(x) = G \\ \Rightarrow \lim_{x\to c} [f(x)+g(x)] = F+G. \end{cases}$$

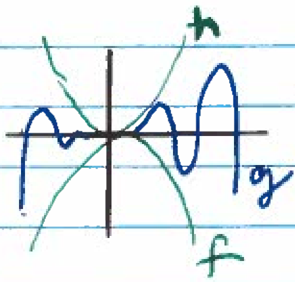
$$\begin{cases} \lim_{x\to c} f(x) = F \text{ and } k\in\mathbb{R} \\ \Rightarrow \lim_{x\to c} [k\cdot f(x)] = K\cdot F \end{cases}$$

Key Examples

- $\lim_{x\to 1} x^2 = 1$
- $\lim_{x\to\infty} \frac{1}{x} = 0$
- $\lim_{x\to 1} 2x+3 = 5$
- $\lim_{x\to 9} \sqrt{x} = 3.$

The Squeeze Theorem

Thm: If $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x)$



then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$.

Pf: Suppose $\underbrace{\lim_{x \to c} f(x)}_{①} = \underbrace{\lim_{x \to c} h(x)}_{②} = L$.

Apply ① to $\varepsilon/2 > 0$ to get:

$$0 < |x - c| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}.$$

Apply ② to $\varepsilon/2 > 0$ to get:

$$0 < |x - c| < \delta_2 \implies |h(x) - L| < \frac{\varepsilon}{2}.$$

We obtain

$$* \quad -\frac{\varepsilon}{2} < f(x) - L < \frac{\varepsilon}{2}$$

$$-\frac{\varepsilon}{2} < h(x) - L < \frac{\varepsilon}{2}. \quad *$$

We get: $|g(x) - L| < \frac{\varepsilon}{2}$ from:

$$-\frac{\varepsilon}{2} < f(x) - L \leq g(x) - L \leq h(x) - L < \frac{\varepsilon}{2}. \quad \text{QED}$$

We continue with one last limit law:

Thm (Products)

If $\underbrace{\lim_{x\to c} f(x) = L}_{①}$ and $\underbrace{\lim_{x\to c} g(x) = M}_{②}$

then $\lim_{x\to c} f(x)g(x) = LM$.

Pf: # Suppose the hypotheses

① $\left[\begin{array}{l}\text{For all } \varepsilon > 0 \\ \quad \text{There is } \delta_1 > 0 : \\ 0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon\end{array}\right.$ ② $\left[\begin{array}{l}\text{For all } \varepsilon > 0 \\ \quad \text{There is } \delta_2 > 0 \\ 0 < |x-c| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon\end{array}\right.$

Main idea

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &\le |f(x)g(x) - Lg(x)| + |Lg(x) - LM| \\ &= \underbrace{|g(x)|}_{\text{BAD (A)}} \cdot \underbrace{|f(x) - L|}_{\text{controlled}} + \underbrace{|L|}_{\text{BAD (B)}}\underbrace{|g(x) - M|}_{\text{controlled}}\end{aligned}$$

To control BAD (A) pick $\delta_A > 0$ so that:

② $\Rightarrow$ $0 < |x-c| < \delta_A \Rightarrow |g(x)| < |M| + 10$ #arbitrary.

To control BAD (B) pick δ_B so that:

② $\Rightarrow$ $0 < |x-c| < \delta_B \Rightarrow |g(x) - M| < \dfrac{\varepsilon}{2 \cdot |L|}$ # $|L| \cdot \dfrac{\varepsilon}{2|L|} = \dfrac{\varepsilon}{2}$

To finish the estimate pick δ_3 so that:

① $\Rightarrow$ $0 < |x-c| < \delta_3 \Rightarrow |f(x) - L| < \dfrac{\varepsilon}{2(|M|+10)}$ # $(|M|+10)\dfrac{\varepsilon}{2(M+10)} = \dfrac{\varepsilon}{2}$

We then take $\delta < \min\{\delta_A, \delta_B, \delta_3\}$

QED.

Defn:

A function $f: \mathbb{R} \to \mathbb{R}$

is CONTINUOUS if:

For all $c \in \mathbb{R}$,

$$\lim_{x \to c} f(x) = f(c)$$

NB: This is the most important property a function can have. It means that we can always approximate the function.

Ex:

- The constant functions are continuous.
- The trig functions $\sin(x)$ and $\cos(x)$ are continuous.

Thm:

Polynomials are all continuous.

Pf:

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ be a polynomial.

$\lim_{x \to c} f(x)$

$= \lim_{x \to c} (a_n x^n + a_{n-1} x^{n-1} + \dots + a_0)$

\# addition

$= \lim_{x \to c} (a_n x^n) + \lim_{x \to c} (a_{n-1} x^{n-1}) + \dots$

\# scaling

$= a_n \lim_{x \to c} x^n + a_{n-1} \lim_{x \to c} x^{n-1} + \dots$

\# multiplication

$= a_n \left[\lim_{x \to c} x\right]^n + a_{n-1} \left[\lim_{x \to c} x\right]^{n-1} + \dots$

\# exercise!

$= a_n \cdot c^n + a_{n-1} c^{n-1} + \dots + a_0$

$= f(c)$

QED.