Thm: If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

then $\lim_{x\to c} (f(x) + g(x)) = L + M$.

Pf: Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$.

We get:

For all $\varepsilon_1 > 0$: # Note the subscript

There is $\delta_1 > 0$:

$0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon_1$

For all $\varepsilon_2 > 0$ # Again, we have a subscript.

There is $\delta_2 > 0$:

$0 < |x-c| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon_2$

Therefore — we fix $\varepsilon > 0$.

Set $\varepsilon_1 = \frac{\varepsilon}{3}$ and $\varepsilon_2 = \frac{\varepsilon}{3}$. # Want: $\varepsilon_1 + \varepsilon_2 < \varepsilon$.

We get δ_1 and δ_2 from the above.

Take $\delta < \min\{\delta_1, \delta_2\}$ # Want: $\delta < \delta_1$
$\delta < \delta_2$

Pf (cont):

If $0 < |x - c| < \delta$ then

$0 < |x-c| < \delta_1$ and $0 < |x-c| < \delta_2$.

Thus,

$$\begin{aligned} &|(f(x)+g(x)) - (L+M)| \\ &= |(f(x)-L) + (g(x)-M)| \\ &\leq \underbrace{|f(x)-L|}_{\frac{\varepsilon}{2}} + \underbrace{|g(x)-M|}_{\frac{\varepsilon}{2}} < \varepsilon. \end{aligned}$$

Therefore,

For all $\varepsilon > 0$:

There is $\delta > 0$:

$0 < |x - c| < \delta \Rightarrow |(f(x)+g(x)) - (L+M)| < \varepsilon.$

Thm: If $f(x) = 2x+3$ then $\lim_{x\to 3} f(x) = 9$.

Pf: Fix $\varepsilon > 0$.

Want: $\delta > 0$ such that
$0 < |x-3| < \delta \Rightarrow |f(x) - 9| < \varepsilon$.

We pick $\delta = \frac{\varepsilon}{2}$

NB: $|(2x+3) - 9|$
$= |2x-6|$
$= 2|x-3|$.

We get:

$$0 < |x-3| < \delta = \frac{\varepsilon}{2}$$

$$\Rightarrow 0 < 2|x-3| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$$

$$\Rightarrow 0 < 2|x-3| < \varepsilon$$ # Almost there!

$$\Rightarrow 0 < |2x-6| < \varepsilon$$

$$\Rightarrow 0 < |(2x+3) - 9| < \varepsilon.$$

QED.

Thm: $\lim_{x \to 1} 10x + 7 = 17$

Pf: Fix $\varepsilon > 0$.

Want $\delta > 0$ such that:

$$0 < |x - 1| < \delta \Rightarrow |(10x + 7) - 17| < \varepsilon.$$

NB: $|(10x + 7) - 17| = |10x - 10|$

$$= 10|x - 1|.$$

Pick $\delta = \frac{\varepsilon}{10}$.

We get:

$$0 < |x - 1| < \delta$$

$$\Rightarrow 0 < |x - 1| < \frac{\varepsilon}{10}$$

$$\Rightarrow 0 < 10|x - 1| < 10\left(\frac{\varepsilon}{10}\right) = \varepsilon$$

$$\Rightarrow 0 < |10x - 10| < \varepsilon$$

$$\Rightarrow 0 < |(10x + 7) - 17| < \varepsilon$$

QED.

Thm: $\lim_{x \to 1} x^2 = 1$.

Pf: Fix $\varepsilon > 0$.

want $\delta > 0$ such that

$$0 < |x-1| < \delta \Rightarrow |x^2 - 1| < \varepsilon.$$

observe: $|(x^2-1)| = \underbrace{\|x-1\|}_{\text{controlled}} \cdot \underbrace{\|x+1\|}_{\text{not controlled}}$

We need to control $\|x+1\|$

If $\delta < \frac{1}{10}$ then $0 < |x-1| < \delta$

$$\Rightarrow 0 < |x-1| < \frac{1}{10}$$

$$\Rightarrow -\frac{1}{10} < x - 1 < \frac{1}{10}$$

$$\Rightarrow 2 - \frac{1}{10} < x + 1 < 2 + \frac{1}{10}$$

$$\Rightarrow |x+1| < \frac{21}{10}$$

Pick $\delta < \min\left\{\frac{1}{10}, \frac{10}{21}\varepsilon\right\}$. ← This controls $\|x+1\|$

$0 < |x-1| < \delta$ [redacted]

$$\Rightarrow 0 < \|x-1\| \cdot \|x+1\| < \delta \|x+1\|$$

$$< \frac{10}{21}\varepsilon \cdot \frac{21}{10} = \varepsilon$$

QED.