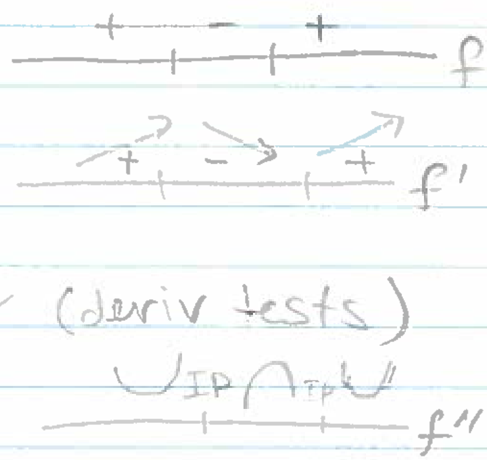
This week: Curve sketching (§3.3)
Optimization (§3.4)

No Related Rates (§3.5)

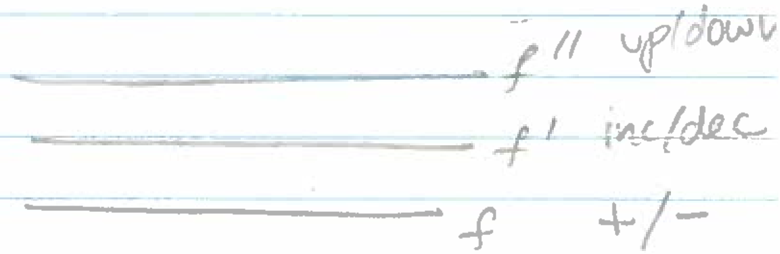
Teaching evaluations! Do them!

The Curve Sketching Algorithm. (p268)

1. Domain
2. Roots of f. Sign chart.
3. f' and critical points.
4. Increasing / Decreasing.
5. Classify critical points
   min, max, neither (deriv tests)
6. Concave up/down



7. Label each extrema or IP by $(c, f(c))$
8. Discontinuities. Non-differentiable points.
9. Limits to points outside domain.
10. $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$
11. Drawing!



A31 Wk 11a

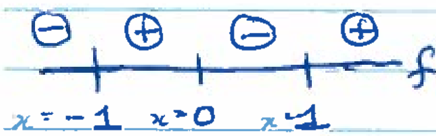
Ex: Sketch the curve $y = f(x) = x^3 - x$

(1) Domain

$$\text{Domain}(f) = \mathbb{R}$$

(2) Roots

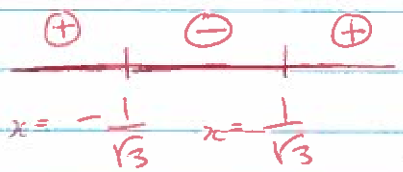
$$f(x) = x^3 - x = x(x^2-1) = x(x-1)(x+1)$$



Thus $x = -1, 0, 1$ are the only roots.

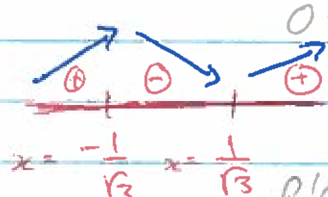
(3) Critical points.

$$f'(x) = 3x^2 - 1$$
$$= (\sqrt{3}x - 1)(\sqrt{3}x + 1)$$



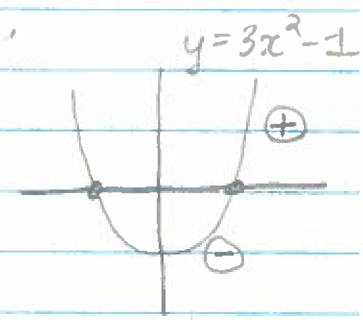
Thus $x = -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ are the only critical points

(4) Increasing/Decreasing



$$0 < f'(x) \iff 0 < 3x^2 - 1$$
$$\iff x < -\frac{1}{\sqrt{3}} \text{ OR } x > \frac{1}{\sqrt{3}}$$

$$f'(x) < 0 \iff 3x^2 - 1 < 0$$
$$\iff -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$



(5) Classify critical points.

Apply 2^{nd} deriv test

$$f''(x) = 6x$$

$f''(-\frac{1}{\sqrt{3}}) < 0 \Rightarrow$ max

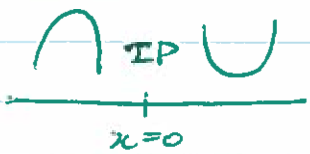
$f''(\frac{1}{\sqrt{3}}) > 0 \Rightarrow$ min

(6) Concave up / down

$f''(x) = 6x \implies$ If $x > 0$ then concave up

If $x < 0$ then concave down

$x = 0$ is an inflection point.



(7) Extrema

$$f\left(-\frac{1}{\sqrt{3}}\right) = \left(-\frac{1}{\sqrt{3}}\right)^3 - \left(-\frac{1}{\sqrt{3}}\right)$$

$$= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{1}{3\sqrt{3}} + \frac{3}{3\sqrt{3}}$$

$$= \frac{2}{3\sqrt{3}}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{2}{3\sqrt{3}}$$

(8) Discontinuities / Non-differentiable

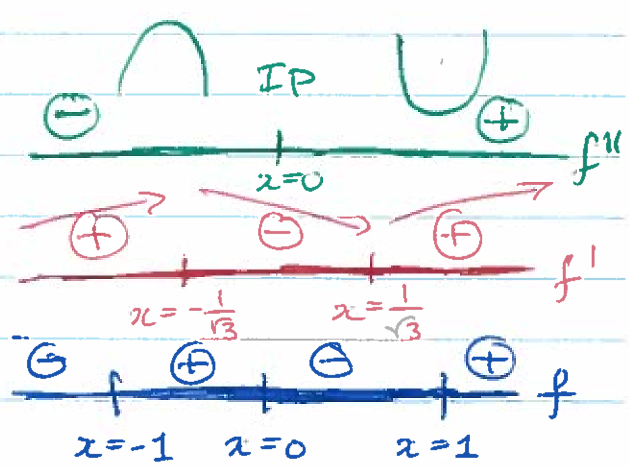
Does not apply.

(9) Limits to points outside domain.

Does not apply.

(10) $\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$



Ex: Sketch the curve $y = f(x) = \dfrac{x}{1-x^2}$

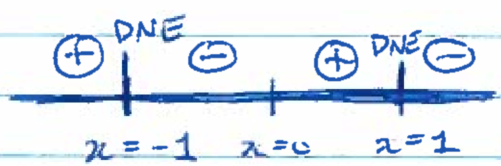
1. Domain

$$\text{Domain}(f) = \{x \in \mathbb{R} : x \neq -1, x \neq 1\}$$

2. Roots

$f(x) = 0 \iff x = 0$ is the only root.

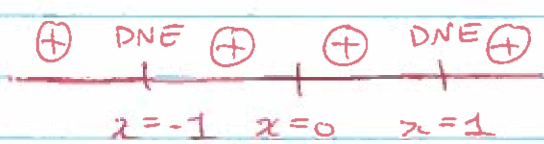
$$0 < f(x) \iff (x > 0 \text{ and } 1-x^2 > 0) \text{ OR } (x < 0 \text{ and } 1-x^2 < 0)$$
$$\iff 0 < x < 1 \qquad\qquad x < -1$$
$$0 > f(x) \iff (x < 0 \text{ and } 1-x^2 > 0) \text{ OR } (x > 0 \text{ and } 1-x^2 < 0)$$
$$\iff -1 < x < 0 \qquad\qquad x > 1$$



3. critical points.

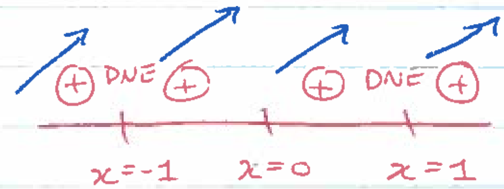
$$f'(x) = \frac{1(1-x^2) - x(-2x)}{(1-x^2)^2}$$
$$= \frac{1+x^2}{(1-x^2)^2}$$

$f'(x) = 0 \iff x = 0$ $\qquad$ $f'(x)$ undef $\iff x = -1$ OR $x = 1$



(4) Increasing / Decreasing

$$f'(x) = \frac{1+x^2}{(1-x^2)^2} \quad \frac{\oplus}{\oplus}$$



(5) Classify critical points.

By the 1st deriv test,
all critical points are neither max nor m..

(6) Concave Up/Down

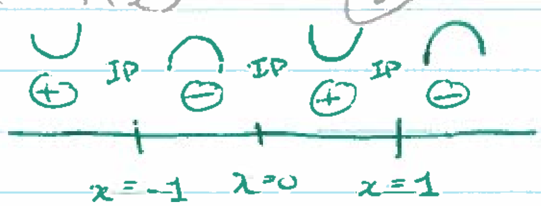
$$f''(x) = \frac{(2x)(1-x^2)^2 - (1+x^2)\cdot[2(1-x^2)\cdot(-2x)]}{(1-x^2)^4}$$

$$= \frac{(1-x^2)[2x(1-x^2) - (1+x^2)\cdot(-4x)]}{(1-x^2)^4}$$

$$= \frac{2x - 2x^3 + 4x + 4x^3}{(1-x^2)^3}$$

$$= \frac{6x + 2x^3}{(1-x^2)^3} = \frac{2x\overset{\oplus}{(3+x^2)}}{(1-x^2)^3}$$

$f''(x) > 0 \Leftrightarrow f(x) > 0$!



7 Extrema

Does not apply. No local min/max.

8 Discontinuities

The function is not defined at $x = -1$ and $x = 1$

9 Limits to points outside domain.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x}{1-x^2} = \frac{\ominus}{\oplus} = -\infty$$

$$\lim_{x \to -1^+} f(x) = \frac{\ominus}{\ominus} = \infty$$

$$\lim_{x \to 1^-} f(x) = \frac{\oplus}{\oplus} = \infty$$

$$\lim_{x \to 1^+} f(x) = \frac{\oplus}{\ominus} = -\infty$$

10 $\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = 0$

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