Recall,

$f: A \to B$ is

one-to-one if:

$f(x) = f(y) \Rightarrow x = y$

onto if:

For all $b \in B$

There is $a \in A$

such that

$f(a) = b$.

Ex: $f: \mathbb{R} \to \mathbb{R}$ given $f(x) = 2x$

Claim: f is injective.

Pf: If $f(x) = f(y)$ then

$2x = 2y$

and thus $x = y$ QED

Claim: f is surjective

Pf: If $y \in \mathbb{R}$ then

$f(\frac{y}{2}) = 2(\frac{y}{2}) = y$.

QED

Ex: $g: \{x \in \mathbb{R} : x \neq 0\} \to \mathbb{R}$

given $g(x) = \frac{1}{x}$

Claim: g is injective

Pf: If $g(x) = g(y)$ then

$\frac{1}{x} = \frac{1}{y}$

thus $\frac{1}{x}(xy) = \frac{1}{y}(xy)$

and $y = x$. QED.

Claim: g is NOT surjective.

Pf: Consider $y = 0 \in \mathbb{R}$.

There is no $x \in \{x \in \mathbb{R} : x \neq 0\}$

such that $\frac{1}{x} = 0$.

Thus $g(x) \neq 0$ for any x.

QED.

Ex: $p: \mathbb{R} \to \mathbb{R}$ given

$p(x) = \pi$ for all x.

Evidently p is

neither one-to-one

nor onto.

<u>Ex</u>: $h : \mathbb{R} \to \mathbb{R}$ given

IN MATH:

$$h(x) = \begin{cases} x+1 & x \leq -1 \\ 0 & -1 < x < 1 \\ x-1 & 1 \leq x \end{cases}$$

IN CODE:

```
func h (x real):
  if (x≤-1):
      return x+1
  else if (-1<x<1):
      return 0
  else if (1≤x):
      return x-1.
```

<u>Claim</u>: h is NOT one-to-one.

<u>Pf</u>: $h(0) = h(\frac{1}{2}) = 0$

and $0 \neq \frac{1}{2}$

Thus, h is not injective.

QED.

Injective \ Surjective	Yes	No
Yes	f	g
No	h	p

<u>Claim</u>: h is surjective.

<u>Pf</u>: Let $y \in \mathbb{R}$ be arbitrary.

We consider three cases:

If $y=0$ then $h(y)=y$. ✓

If $y>0$ then $h(y+1)=(y+1)-1=y$. ✓

If $y<0$ then $h(y-1)=(y-1)+1=y$. ✓

Therefore h is surjective.

Functions and Graphs

Def^n:

The GRAPH of $f(x)$ is:

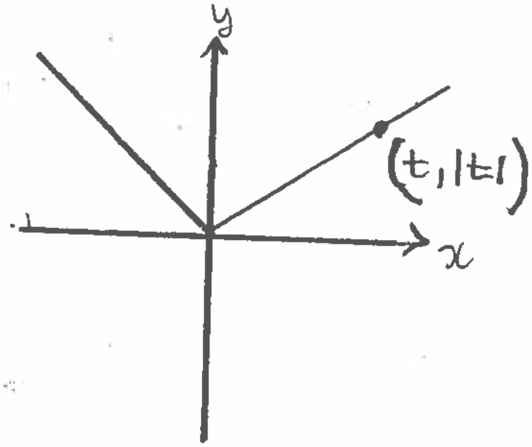
$$\text{Graph}(f) = \{ (x, f(x)) : x \in \text{Domain}(f) \}$$

NB: The graph is a subset of the plane $\mathbb{R}^2$.

We will use calculus to study the geometry of Graph(f).

Ex: The graph of $a(x) = |x|$.

Recall, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



Transformations

Q: Suppose $f: \mathbb{R} \to \mathbb{R}$ and $g(x) = f(x) + c$.

How are graph(f) and Graph(g) related?

Ans:

Graph(g) is obtained by moving Graph(f)

UP if $c > 0$

DOWN if $c < 0$

Q: Suppose $g(x) = f(x + c)$

Ans: Graph(g) is obtained by moving Graph(f)

LEFT if $c > 0$

RIGHT if $c < 0$.

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Transformations (con't)

2: Suppose $g(x) = cf(x)$

ns:

VERTICALLY STRETCH	$\lvert c\rvert > 1$
VERTICALLY COMPRESS	$\lvert c\rvert < 1$
VERTICALLY REFLECT	$c < 0.$

3: Suppose $g(x) = f(c \cdot x)$

HORIZONTALLY COMPRESS	$\lvert c\rvert > 1$
HORIZONTALLY STRETCH	$\lvert c\rvert < 1$
HORIZONTALLY REFLECT	$c < 0$

In summary,

We can analyze

$$g(x) = Af(Bx+C) + D$$

for any A, B, C, D.

This gives a very versatile method for adjusting or fine-tuning a function.

Composition

Def^n: Let $g: A \to B$ and $f: B \to C$

The COMPOSITION of f and g is

$$h(x) = (f \circ g)(x) = f(g(x))$$

Inverses:

Def^n: The INVERSE of a function $f: A \to B$ is a function $g: B \to A$ such that:

$$(g \circ f)(a) = a$$
$$(f \circ g)(b) = b$$

for all $a \in A$ and $b \in B$.

Ex: What is the inverse function of $f(x) = 2x+1$?

Find a value $g(x)$ such that $f(g(x)) = x$

Suppose $y = 2x+1$

Solve for x.

$$x = \frac{y-1}{2}$$

Let $g(x) = \dfrac{x-1}{2}$.

Check $f(g(x)) = x$.

Ex (con't)

$$f(g(x)) = f\left(\frac{x-1}{2}\right)$$
$$= 2\left(\frac{x-1}{2}\right) + 1$$
$$= (x-1)+1 = x$$

Check $g(f(x)) = x$.

$$g(f(x)) = g(2x+1)$$
$$= \frac{(2x+1)-1}{2}$$
$$= \frac{2x}{2} = x.$$

Ex: Does the constant function $p(x) = \pi$ have an inverse?

No — There is no function $g(x)$ such that

$$p(g(x)) = x$$

for all x. (Any $x \neq \pi$

Inequalities / Intervals.

Defn: "a is less than b"

$a < b$ is a STRICT inequality

$a \leq b$ is a WEAK inequality

Defn: Intervals.

$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$

$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$

* This is NOT a point!

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Defn: Distance

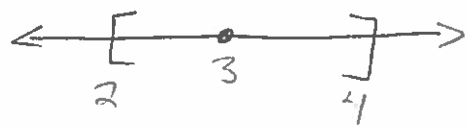
The distance between $a, b \in \mathbb{R}$ is

$$d(a,b) = |a-b|$$

Ex: Write the set of points that are at most one unit away from the number 3.

Find all x such that $d(x,3) \leq 1$

Solve $|x-3| \leq 1$.



Thus, $[2,4]$ is the desired interval.

Ex: $d(x,5) < \varepsilon$

$$(5-\varepsilon, 5+\varepsilon)$$

The notion of distance is very important.