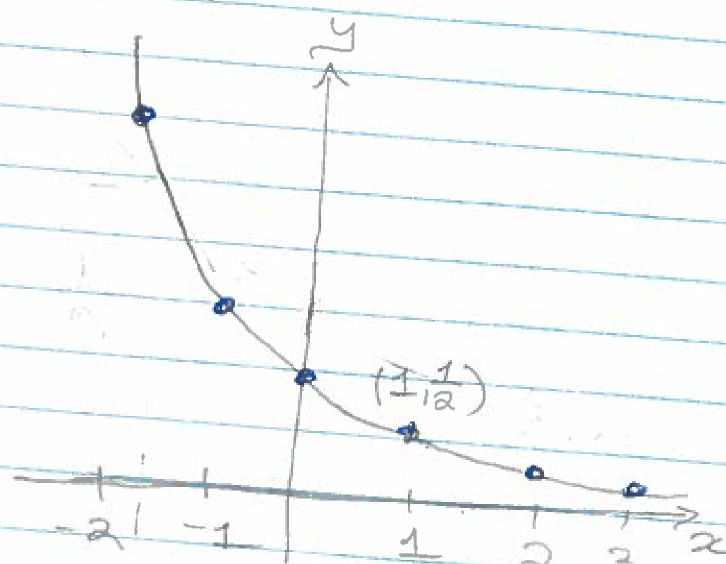
A29 Wk 8a

This week § 4.1 – 4.3

Ex: Sketch the graph of $y = (\frac{1}{2})^x$

Make a table of values

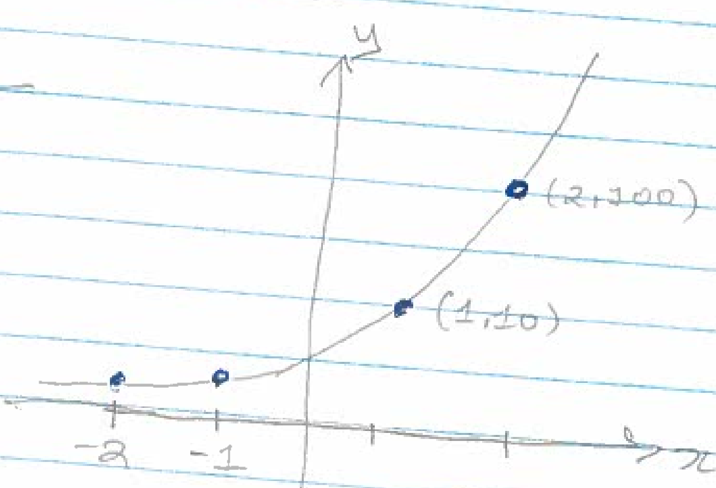
x	$y = (\frac{1}{2})^x$
3	1/8
2	1/4
1	1/2
0	1
−1	2
−2	4



Ex. Sketch the graph of $y = 10^x$

Make a table of values

x	$y = 10^x$
3	1000
2	100
1	10
0	1
−1	1/10
−2	1/100



Defn: An EXPONENTIAL FUNCTION is of the form $y = a^x$ where $a > 0$ and $a \neq 1$. The number a is the BASE.

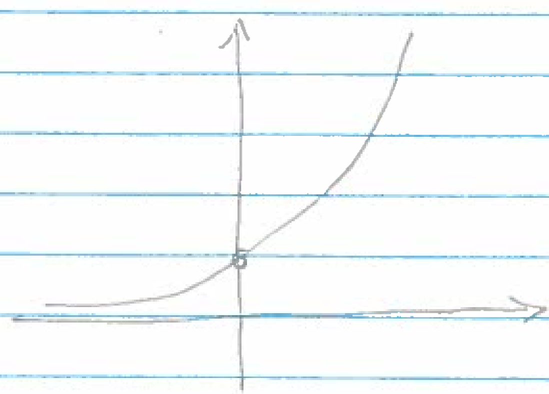
Fact:
① $a^x a^y = a^{(x+y)}$
② $a^x / a^y = a^{(x-y)}$
③ $(a^x)^y = a^{(xy)}$
④ $a^{-x} = 1/a^x$

Ex: $10^2 \cdot 10^3 = 100 \cdot 1000 = 100000 = 10^5$

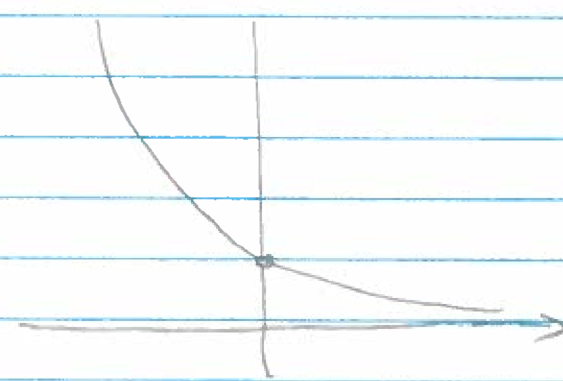
$\frac{10^3}{10^2} = \frac{1000}{100} = 10 = 10^{3-1}$

$(10^2)^3 = 100 \cdot 100 \cdot 100 = 1000000 = 10^6$

Defn:



EXPONENTIAL GROWTH
$a > 1$



EXPONENTIAL DECAY
$a < 1$

Ex: Suppose you have a population of bacteria in a petri dish. The population is observed to grow 10% every hour.

If there are 100 bacteria in the culture initially, how many bacteria will there be in 4 hours?

$$P(t) = 100\left(1+\frac{1}{10}\right)^t$$

Ex: Use a calculator to make a table of values for P(t).

t	$P(t)$	$P(t+1)-P(t)$
0	100	
1	110	10
2	121	11
3	133.1	12.1
4	146.41	13.31
5	161.51	15.10
6	171.15	16.10

NB: The differences keep getting larger. Thus, the rate of change keeps going up and up.

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Ex: Find the average rate of change for $y = 2^x$ between $x=0$ and $x = 1/10$

$f(x) = 2^x$
$\Rightarrow f'(0) < 1.$

$$m = \frac{2^{1/10} - 2^0}{1/10 - 0} = 0.71773$$

Ex: Find the average rate of change for $y = 3^x$ between $x=0$ and $x = 1/10$

$g(x) = 3^x$
$\Rightarrow g'(0) > 1$

$$m = \frac{3^{1/10} - 3^0}{1/10 - 0} = 1.16123$$

Defn: There is a unique number

$$e \simeq 2.718$$

such that: $f(x) = e^x \Rightarrow f'(0) = 1$

Facts: ① $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

② $\lim_{x \to 0} \frac{e^x - 1}{x} = 0$

③ $\frac{d}{dx} e^x = e^x$ Whoa!

④ The notation e^x for exponentiation pioneered by Euler. (oiler)

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Ex: Calculate the derivative of $y = e^{-x^2}$.

$$\frac{dy}{dx} = \frac{de^{-x^2}}{dx}$$

$$= \frac{de^{-x^2}}{d(-x^2)} \cdot \frac{d(-x^2)}{dx} \quad \text{\# chain rule}$$

$$= e^{-x^2} \cdot (-2x)$$

Ex: Calculate the derivative of $y = \frac{e^{3x}}{x}$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(e^{3x}) \cdot x - e^{3x} \frac{d}{dx}(x)}{x^2} \quad \text{\# quotient}$$

$$= \frac{3e^{3x} \cdot x - e^{3x} \cdot 1}{x^2} \quad \text{\# chain}$$

Summary:

- Exponential growth models populations growing.
- $y = a^x$ $(a > 1)$ exponential growth
- $y = a^x$ $(a < 1)$ exponential decay
- $\frac{d}{dx}(e^x) = e^x$
- $e \simeq 2.71$ is a number.

Logarithms

Ex: Solve $10^x = 10$.
$10^y = 100$.
$10^z = 1000$.

The values $x=1$, $y=2$, $z=3$ work.

Ex: Solve $2^x = 256$
$2^y = 512$
$2^z = 2048$

The values $x=8$, $y=9$, $z=11$ work.

Ex: Does $3^t = 78$ have whole number solution?

$3^3 = 27$ NB: $3^3 \neq 9$. :-)

$3^4 = 81$.

Therefore, $3^t = 78$ does not have a whole number solution.

NB: $3^t = 78$ has some solution $3 < t < 4$.

See: Anki, Memnosyne, SSRs.

We define the solution to $3^t = 78$ to be

$$t = \log_3 78.$$

Defn: If $a^t = b$ then $t = \log_a(b)$ for $a > 0$.

"the base a logarithm of b"

Ex: $\log_{10}(1000) = 3$, $\log_2(256) = 8$, $\log_a(a^7) = 7$.

Notation: $\underbrace{\log(x) = \log_{10}(x)}_{\text{"common log"}}$ and $\underbrace{\ln(x) = \log_e(x)}_{\text{"lawn } x\text{" = "natural log"}}$.

Fact:
- $\log_a(a^x) = x$
- $a^{\log_a(x)} = x$

$\log_a(x)$ is the inverse of a^x.

- $\log_a(1) = 0$
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(x^k) = k\log_a(x)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

Exercise: Convince your friends of these facts.

Logarithms were invented/discovered by John Napier ~1614.
They were made to simplify calculation.

<u>Fact</u> Consider two N digit numbers.
Addition requires ~N steps.
Multiplication ~N^2 steps.

Using a table of logarithms multiplication requires ~N steps.

<u>Ex</u>: Given $\log_{10}(231) = 2.36311$

$\log_{10}(597) = 6.39191$

Calculate $\log_{10}(231 \cdot 597) = 8.75502$

<u>Ex</u>: If $\log_A(2) = X$ and $\log_A(3) = Y$

What is $\log_A(108)$?

$108 = 2^2 3^3$

$\Rightarrow \log_A(108) = \log_A(2^2 \cdot 3^3)$

$= \log_A(2^2) + \log_A(3^3)$

$= 2X + 3Y.$

Thm: $\frac{d}{dx} \ln(x) = \frac{1}{x}$

$$e^{\ln(x)} = x \longrightarrow \frac{d}{dx}\left[e^{\ln(x)}\right] = \frac{d}{dx}[x]$$

$$\Rightarrow \frac{de^{\ln(x)}}{d\ln(x)} \cdot \frac{d\ln(x)}{dx} = 1$$

Whoa! Magic.

$$\Rightarrow e^{\ln(x)} \cdot \frac{d\ln(x)}{dx} = 1$$

$$\Rightarrow x \frac{d\ln(x)}{dx} = 1$$

$$\Rightarrow \frac{d}{dx}[\ln(x)] = \frac{1}{x}.$$

Ex: Find the derivative of $y = \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{d\ln(x^2+1)}{d(x^2+1)} \frac{d(x^2+1)}{dx}$$ # chain rule

$$= \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}.$$

Ex: Find the derivative of $y = x\ln(x^2+1)$

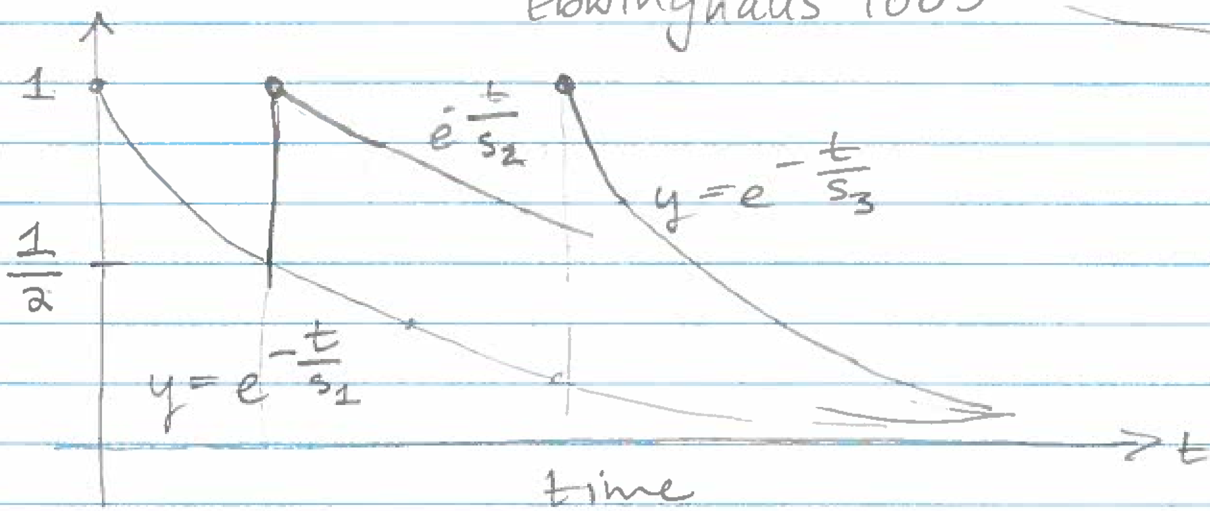
$$\frac{dy}{dx} = \frac{d}{dx}[x]\ln(x+1) + x\frac{d}{dx}\ln(x+1)$$ # product

$$= \ln(x+1) + \frac{x}{x+1}.$$

The Ebbinghaus Learning Model (p289)

[wikipedia: The Forgetting Curve]

Ebbinghaus 1885



The probability of retaining a memory of strength s after time t is $\approx e^{-\frac{t}{s}}$.

Repeating a memory changes its strength.

To maximize the change, study when $\mathbb{P} = \frac{1}{2}$.

$(S_1 < S_2)$

If you study after $\mathbb{P} = \frac{1}{2}$ you get $S_1 \approx S_3$.