MAT A29 — Wk 7a

This week we will look at: §3.5 – 7

Today we continue with optimization.

Ex: (Maximize revenue) p232

You own a movie theater. Trials show:
If you charge \$6 admission, then 150 attend.
For every \$1 increase in cost, 20 people leave.

Also — People spend \$1.80 on snacks!

How much should you charge?

State the problem mathematically.

Let $R(x)$ = revenue with ticket price x.

$$= (\text{revenue from tickets}) + (\text{rev. snacks})$$

$$= \underbrace{(150-20x)}_{\text{people}}\underbrace{(6+x)}_{\text{tickets}} + \underbrace{1.80}_{\text{snacks}}\underbrace{(150-20x)}_{\text{people}}$$

$$= -20x^2 - 6x + 1170$$

$$= 900 + 150x - 120x - 20x^2 + 270 - 36x$$

We want to maximize $R(x)$

Ex (con't):

\# To maximize/minimize $R(x)$
\# Find critical points
\# Solve $R'(x) = 0$!

$$R'(x) = -20 \cdot 2x - 6$$
$$= -40x - 6$$

$$R'(x) = 0 \Rightarrow x = -\frac{6}{40} = \frac{3}{20} = -0.15$$

Thus $\cancel{x} = -0.15$ is a critical point of $R(x)$.

\# Check that $x = 0.15$ is a max.
\# Apply the 2nd derivative test for rel. extrema.

$$R''(x) = -40 \Rightarrow R''(0.15) = -40$$

Thus, $x = 0.15$ is a maximum.

It follows that revenue is maximized when we have a ticket price of:

$\$6 - \$0.15 = \$5.85$

Hidden assumptions?

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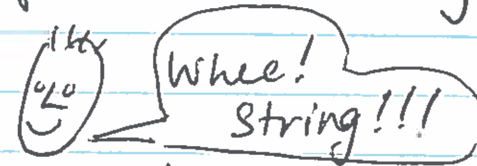
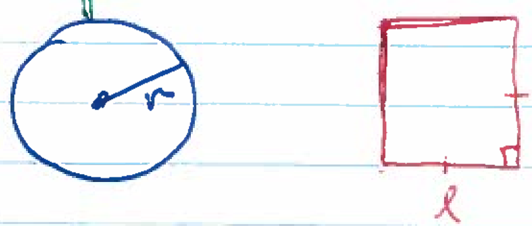
Ex: You have a 24 cm long piece of string and you wish to make a circle and square of the largest possible total area.



What do you do?

Draw a picture. Label the variables



Formulate the problem mathematically.

$$2\pi r + 4l = 24 \text{ cm}$$

$$A(r) = \pi r^2 \qquad A(l) = l^2$$

Note the dependence of variables
If we have r we get l and vice-versa.

$$l = \frac{24 - 2\pi r}{4} = 6 - \left(\frac{\pi}{2}\right) r$$

Let the total area be: $T(r) = \pi r^2 + \left(6 - \left(\frac{\pi}{2}\right) r\right)^2$

[Maximize $T(r)$

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Ex (con't):

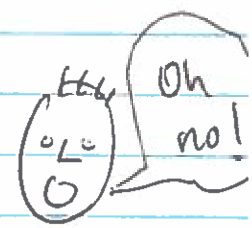
\# To maximize/minimize $T(r)$
\# Find critical points
\# Solve $T'(r)=0$

$$T'(r) = 2\pi r + 2\left(-\frac{\pi}{2}\right)\left[6-\left(\frac{\pi}{2}\right)r\right]$$
$$= \left(2\pi + \frac{\pi^2}{2}\right)r - 6\pi.$$

$$T'(r)=0 \Rightarrow r = \frac{6\pi}{2\pi+\frac{\pi^2}{2}} = \frac{12}{4+\pi}$$

\# Check that $x = \frac{12}{4+\pi}$ is a maximum.
\# Apply 2nd deriv test.

$$T''(r) = 2\pi + \frac{\pi^2}{2} > 0$$



Thus, $x = \frac{12}{4+\pi}$ is actually a minimum.

$$A(0) = 6^2 = 36\,\text{cm}^2$$
$$A\left(\frac{24}{2\pi}\right) = \pi\left(\frac{24}{2\pi}\right)^2 = \frac{144}{\pi} \cong 45.8\,\text{cm}^2$$

Thus, we should make the whole string a circle.

End Points

Ex: Maximize $f(x)=x^2$ for $1 \le x \le 3$.

The usual technique of looking for critical points does not work in this case because $f(x)$ has one critical point $x=0$ and it is not in the allowed range.

Thus, we check the endpoints.

$$f(1)=1^2=1 \quad \text{and} \quad f(3)=3^2=9.$$

Therefore $x=3$ is the maximizer for $1 \le x \le 3$

Summary

To solve a maximization/minimization word problem:

1. Draw an accurate labelled diagram
2. State the problem mathematically
3. Find critical points
4. Apply 2nd deriv test
5. Check the endpoints.

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Implicit Differentiation

So far — we can only differentiate variables given to us in the form:

$$y = f(x)$$

This is called an explicit relationship.

Ex: Differentiate y with respect to x if:

$$x^2 + y^2 = 1.$$

Implicit relationship
No isolated variables.

Solution # 1:

Isolate for y as a function of x.

$$y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

$$= -\frac{x}{y}$$

Solution # 2:

Suppose y is a function of x.
Let $y = f(x)$

$$x^2 + y^2 = 1 \Rightarrow x^2 + [f(x)]^2 = 1$$

$$\Rightarrow 2x + 2f(x) \cdot f'(x) = 0$$

$$\Rightarrow f'(x) = -\frac{2x}{2f(x)} = -\frac{x}{f(x)} = -\frac{x}{y}$$

Solution #2 is very useful since it lets us handle equations that we cannot solve explicity.

Ex: Find $\frac{dy}{dx}$ if: $\cos(y^2) + xy = 0$.

Suppose $y = f(x)$ and work out the derivative

$$\cos([f(x)]^2) + xf(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\cos([f(x)]^2) + xf(x)\right) = \frac{d}{dx}(0)$$

$$\Rightarrow -2f(x)f'(x)\sin([f(x)]^2) + f(x) + xf'(x) = 0$$

$$\Rightarrow f'(x)\left[-2f(x)\sin([f(x)]^2) + x\right] = -f(x)$$

$$\Rightarrow f'(x) = \frac{-f(x)}{-2f(x)\sin([f(x)]^2) + x}$$

Undo the Substitution.

$$\frac{dy}{dx} = \frac{-y}{-2y\sin(y^2) + x}$$

Plan: 1 Let $y = f(x)$
2 Differentiate
3 Replace $f(x)$ with y and $f'(x)$ with $\frac{dy}{dx}$

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Related Rates

Ex: The surface area of a circular puddle is increasing at a rate of $\pi\ cm^2/s$

How fast is its radius increasing when the radius $= 10cm$?

Name the variables

Find a relationship between the variables.

Let A = surface area of puddle in cm^2
r = radius of puddle in cm.

$$A = \pi r^2$$ ← NB: Both depend on time!

We get $\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$ The chain Rule!

Using the given data:

$$\pi = 2\pi \cdot 10 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{\pi}{2\pi \cdot 10} = \frac{1}{20}\ cm/s.$$

Thus, the radius is increasing at a rate of $\frac{1}{20}$ cm/s when $r = 10cm$

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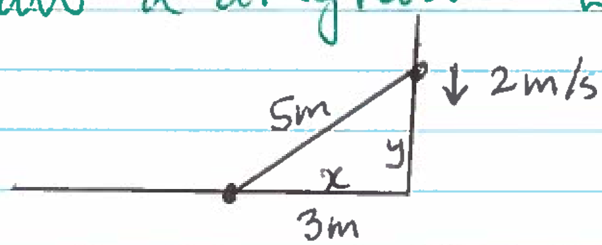
Ex: A 5m tall ladder is leaning against a wall.

It starts to slip.

Suppose the top is falling at 3m/s.

How fast is the bottom slipping when it is 3m away?
(from the wall)

Draw a diagram. Label the variables



$x^2+y^2=5^2$

$3^2+y^2=5^2$

$\Longrightarrow \quad y=4.$

Find a relationship among vars.

$$x^2+y^2=5^2$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

Down $\Rightarrow \frac{dy}{dt}$ $\ominus$

$$2\cdot 3\frac{dx}{dx}+2\cdot 4\cdot(-2)=0$$

$$\frac{dx}{dt}=\frac{16}{6}=\frac{8}{3}\ \text{m/s}.$$