MAT A29 Wk 5a

This week: §3.1 and 3.2

Optimization and curve sketching.

"Calculus studies rates of change, it can help us find large/small values."

Def^n: A function $f:\mathbb{R}\to\mathbb{R}$ is INCREASING if

$$a<b \Rightarrow f(a)<f(b)$$

Def^n: A function $f:\mathbb{R}\to\mathbb{R}$ is DECREASING if

$$a<b \Rightarrow f(b)<f(a)$$

Ex: The function $f(x)=10x$ is increasing.

Ex: The function $g(x)=-3x$ is decreasing.

$-$ slope $\Rightarrow$ decrease

$+$ slope $\Rightarrow$ increasing

Fact:
- If $f'(x)>0$ for all $x\in(a,b)$ then f is increasing on (a,b).
- If $f'(x)<0$ for all $x\in(a,b)$ then f is decreasing on (a,b)

Ex: Consider $f(x)=x^2$. Where is f increasing? dec?

$f'(x)=2x \Rightarrow$ f is increasing on $(0,\infty)$
f is decreasing on $(-\infty,0)$

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Def^n: $x=c$ is a CRITICAL POINT of $f(x)$

If: $\begin{cases} f'(x) \text{ is not defined at } x=c \\ \text{OR} \\ f'(c)=0 \end{cases}$

Ex: Find the critical points of $f(x)=\sqrt{|x|}-x^2$.

write the function explicitly $f(x)=\begin{cases} \sqrt{x}-x^2, & x>0 \\ \sqrt{-x}-x^2, & x\leq 0 \end{cases}$

compute derivatives of both parts.

If $x>0$ then $f(x)=\sqrt{x}-x^2$ and

$$f'(x)=\frac{1}{2\sqrt{x}}-2x$$

If $x\leq 0$ then $f(x)=\sqrt{-x}-x^2$

$$f'(x)=-\frac{1}{2\sqrt{-x}}-2x$$

! $x\leq 0$ means the $\ominus$ sign is okay.

Determine where $f'(x)$ is undefined.

$x=0$ is a critical value because $f'(x)$ is undefined at $x=0$.

Determine where $f'(x)=0$.

$f'(x)=0$ and $x>0 \Rightarrow 0=\frac{1}{2\sqrt{x}}-2x$

$\Rightarrow 0=1-4x^{\frac{3}{2}}$

$\Rightarrow \frac{1}{4}=x^{\frac{3}{2}} \Rightarrow x=\left(\frac{1}{4}\right)^{\frac{2}{3}}$

Similarly, $f'(x)=0$ and $x\leq 0$ $\Rightarrow x=-\left(\frac{1}{4}\right)^{\frac{2}{3}}$

Thus, $x=0, \left(\frac{1}{4}\right)^{\frac{2}{3}}, -\left(\frac{1}{4}\right)^{\frac{2}{3}}$ are critical values.

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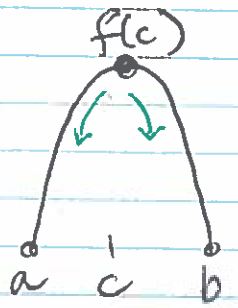
Def^n: $x = c$ is a (RELATIVE) MAXIMUM if

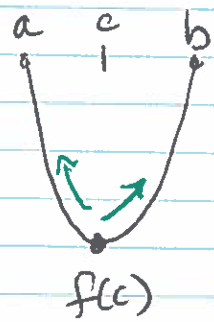


There is $(a,b) = \{x : a < x < b\}$

such that: $c \in (a,b)$ and

$x \in (a,b) \Rightarrow f(x) \leq f(c)$.

Def^n: $x = c$ is a (RELATIVE) [redacted] MINIMUM if



There is (a,b) such that: $c \in (a,b)$

and $x \in (a,b) \Rightarrow f(c) \leq f(x)$.

\# f(c) is smaller than its neighbours.

These are called "relative extrema"

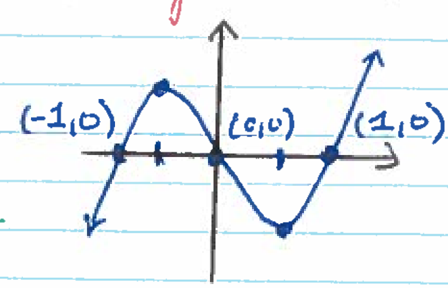
Thm: If $x = c$ is a relative max or relative min then $x = c$ is a critical point of f.

NB: If extreme value then critical value

This means, to find extreme values we need to check critical points.

Ex: Find the extreme values of:

$$f(x) = x(x-1)(x+1)$$
$$= x^3 - x$$



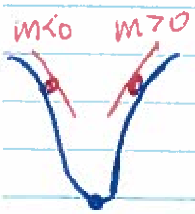
\# Find where $f'(x) = 0$ OR UNDEF.

$$f'(x) = 3x^2 - 1 = 0$$
$$\Longrightarrow x = \pm \frac{1}{\sqrt{3}}$$

Thm (First Derivative Test) p166

Suppose f has a unique critical point
$$a < x = c < b$$

$\boxed{F1}$ If $f'(x) < 0$ on (a,c) and
$f'(x) > 0$ on (c,b)

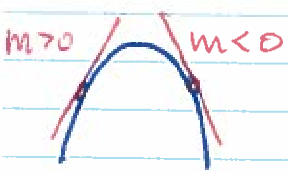


then $x = c$ is a relative minimum.

#


$\Longrightarrow$ relative min.

$\boxed{F2}$ If $f'(x) > 0$ on (a,c) and
$f'(x) < 0$ on (c,b)

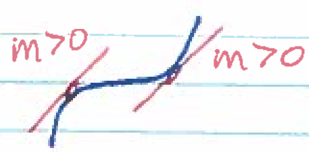


then $x = c$ is a relative maximum.

#


$\Longrightarrow$ relative max.

$\boxed{F3}$ If f' has the same sign on (a,c) and (c,b)
then $x = c$ is neither max nor min.

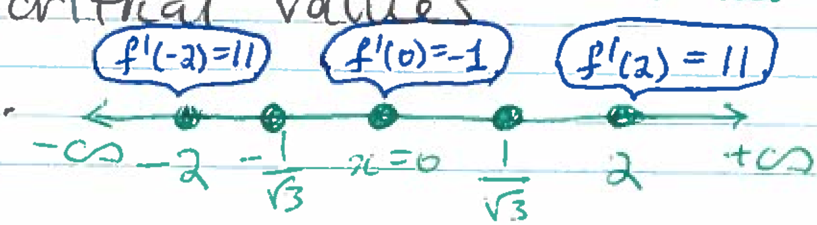


Ex: Classify the critical values of $f(x) = x^3 - x$.

Recall, $x = -\frac{1}{\sqrt{3}}$ and $x = +\frac{1}{\sqrt{3}}$

are the only critical values. # Select test values.

$f'(x) = 3x^2 - 1.$



The Curve Sketching Table (v1)

$x=-\frac{1}{\sqrt{3}}$ $x=\frac{1}{\sqrt{3}}$ Column dividers for critical values.

Interval	$\left(-\infty, -\frac{1}{\sqrt{3}}\right)$	$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\left(\frac{1}{\sqrt{3}}, \infty\right)$
Test value	$x=-2$	$x=0$	$x=3$
sign $f'(x)$	$+$	$-$	$+$
Behavior	f increasing ↗	f decreasing ↘	f increasing ↗

chosen from inside interval.

Result: rel. max rel. min.

To generate a curve sketching table.

1. Find critical values.
   - 1a. Where is $f'(x)$ undefined?
   - 1b. Where is $f'(x)=0$?
2. Create dividers for each crit value.
3. Write intervals between crit values
4. Select test values from intervals
   Pick any nice value!
5. Calculate $f'(x)$ at test values.

See pg 168

<u>Def^n</u>: The FIRST DERIVATIVE of $f(x)$ is $f'(x)$.

The SECOND DERIVATIVE of $f(x)$ is $f''(x) = \frac{d}{dx} f'(x)$

etc.

<u>Ex</u>: Compute the first three derivatives of $f(x) = x^3 - x$.

\# First

$$f'(x) = 3x^2 - 1$$

\# Second

$$f''(x) = 6x$$

\# Third

$$f'''(x) = 6.$$

<u>Def^n</u>: f is CONCAVE UP on (a,b) if:

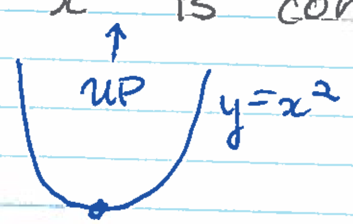
$$x \in (a,b) \Rightarrow f''(x) > 0$$

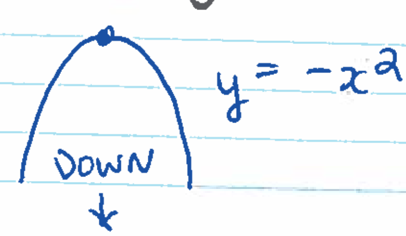
f is CONCAVE DOWN if:

$$x \in (a,b) \Rightarrow f''(x) < 0.$$

<u>Ex</u>: $f(x) = x^2$ is concave up. $f''(x) = 2 > 0$.

$g(x) = -x^2$ is concave down $g''(x) = -2 < 0$.





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Thm (Second Deriv. Test) p180

Suppose $f'(x)$ exists for every $x \in (a,b)$ and $f'(c)=0$ for some $c \in (a,b)$.

1. If $f''(c) > 0$ then $f(c)$ is a rel. min.
2. If $f''(c) < 0$ then $f(c)$ is a rel. max.
3. If $f''(c) = 0$ then no conclusion.

Ex: Find and classify the critical values of

$$f(x) = x^4 - 2x^2$$

compute $f'(x)$ and $f''(x)$.

$$f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$$

$$f''(x) = 12x^2 - 4 = 12\left(x^2 - \frac{1}{3}\right) = 12\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right)$$

Find crit. values. Solve $f'(x)=0$.

$$x = -1, 0, 1.$$

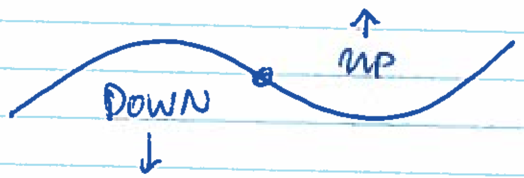
Determine concavity at crit values

$$f''(-1) = 12(-1)^2 - 4 = 8 > 0 \Rightarrow \text{rel. min.}$$
$$f''(0) = 12 \cdot 0^2 - 4 = -4 < 0 \Rightarrow \text{rel. max.}$$
$$f''(1) = 12(1)^2 - 4 = 8 > 0 \Rightarrow \text{rel. min.}$$

Thus $x=-1$ and $x=1$ are relative min/ma.

$x=0$ is a relative maximum.

Defn: $x=c$ is a POINT OF INFLECTION if $f(x)$ has different concavity on either side of $x=c$.



Thm: If $f(x)$ has a point of inflection at $x=c$ then

$$f''(c)=0 \quad \text{OR} \quad f''(x) \text{ is not defined at } x=c$$

Ex: Find the points of inflection of

$$f(x) = 3x^5 - 20x^3$$

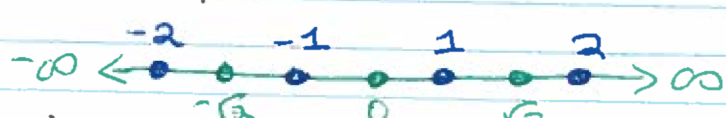
Solve $f''(x)=0$.
compute $f'(x)$ and $f''(x)$

$$f'(x) = 15x^4 - 60x^2 = 15x^2(x^2-4) = 15x^2(x-2)(x+2)$$

$$f''(x) = 60x^3 - 120x = 60x(x^2-2) = 60x(x-\sqrt{2})(x+\sqrt{2})$$

$$f''(x) = 0 \Longrightarrow x = -\sqrt{2}, 0, \sqrt{2}.$$

Check concavity



$f''(-2) = -$ $\qquad$ $f''(1) = -$

$f''(-1) = +$ $\qquad$ $f''(2) = +$

MAT A29 Wk 5b

<u>Ex</u>: If $f(x) = x^3 + 3x^2 - 9x - 13$

then make a Curve Sketching Table (v1) for f and classify the concavity of $f(x)$.

Find critical points.

$$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3)$$
$$= 3(x+3)(x-1)$$

$x = -3$ $\quad$ $x = 1$

# interval	$(-\infty, -3)$	$(-3, 1)$	$(1, \infty)$
# test value	$x = -4$	$x = 0$	$x = 2$
# inc./dec.	$f'(-4) = 15 \nearrow$	$f'(0) = -9 \searrow$	$f'(2) = 15 \nearrow$

Result: $x = -3$ rel max $\quad$ $x = 1$ rel min

To determine concavity, compute $f''(x)$

$$f''(x) = 6x + 6$$

$f''(x) > 0 \iff x > -1.$

$f''(x) < 0 \iff x < -1.$

Thus,

f has a relative max at $x = -3$

min $\quad x = 1$

is concave up on $(-1, \infty)$

down on $(-\infty, -1)$.

has a point of inflection at $x = -1$.

Curve sketching (§3.3)

Defn: A rational function is

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials.

Ex: Find $\lim_{x \to \infty} \frac{x^2+1}{3x^2+x+1}$

Examine some values

$$x = 1 \implies \frac{1^2+1}{3\cdot 1^2+1+1} = \frac{2}{5}$$

$$x = 10 \implies \frac{10^2+1}{3\cdot 10^2+10+1} = \frac{101}{311} \sim \frac{1}{3}$$

$$x = 100 \implies \frac{100^2+1}{3\cdot 100^2+100+1} = \frac{10001}{30101} \sim \frac{1}{3}$$

Work the limit

$$\lim_{x \to \infty} \frac{x^2+1}{3x^2+x+1}$$

These terms get very small.

$$= \lim_{x \to \infty} \frac{1+\frac{1}{x^2}}{3+\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{3}$$

Fact: $\lim_{x\to\infty} \frac{a}{b\cdot x^n} = 0$ for $n>0$ $(b\neq 0)$

Ex: Find $\lim_{x\to\infty} \frac{x^2+3x}{x^5+x^4+10}$

$$= \lim_{x\to\infty} \frac{x^2+3x}{x^5+x^4+10}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)$$

$$= \lim_{x\to\infty} \frac{1+\frac{3}{x}}{x^3+x^2+\frac{10}{x^2}}$$

These become very small.

$$= 0.$$

Observation:

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function.

- If $\deg(P) = \deg(Q)$ then $\lim_{x\to\infty} f(x)$ exists and is finite.
- If $\deg(P) < \deg(Q)$ then $\lim_{x\to\infty} f(x) = 0$.
- If $\deg(P) > \deg(Q)$ then $\lim_{x\to\infty} f(x) = \infty$.

Test this on your own!

Defn: $x=a$ is a VERTICAL ASYMPTOTE if: any of the following limits holds:

$$\lim_{x\to a^-} f(x)=\infty$$

$$\lim_{x\to a^-} f(x)=-\infty \qquad \boxed{\lim_{x\to a^\pm} f(x)=\pm\infty}$$

$$\lim_{x\to a^+} f(x)=\infty$$

$$\lim_{x\to a^+} f(x)=-\infty$$

Ex: Find the vertical asymptotes of $f(x)=\dfrac{1}{x^2-5x+6}$

We note that

$$f(x)=\frac{1}{x^2-5x+6}=\frac{1}{(x-2)(x-3)}$$

Thus $x=2$ and $x=3$ are vertical asymptotes of $f(x)$.

Defn: $y=b$ is a HORIZONTAL ASYMPTOTE if:

$$\lim_{x\to\infty} f(x)=b \quad \text{OR} \quad \lim_{x\to-\infty} f(x)=b$$

Defn: $y = mx + b$ is an OBLIQUE ASYMPTOTE of $f(x) = \dfrac{P(x)}{Q(x)}$ if:

deg(P(x)) = deg(Q(x)) + 1 needs to hold.

$$f(x) = (mx+b) + g(x)$$

where $\lim_{x \to \infty} g(x) = 0$ and $\lim_{x \to -\infty} g(x) = 0$.

Ex: Find the oblique asymptote of

$$f(x) = \frac{x^2 - 3}{x - 1}.$$

Introduce $(x-1)$ in to the top.

$$f(x) = \frac{x^2 - 1 + 1 - 3}{x - 1}$$

$$= \frac{(x^2 - 1) - 2}{x - 1}$$

$$= \frac{(x-1)(x+1) - 2}{x - 1}$$

$$= (x+1) - \frac{2}{x-1}$$

Review: polynomial long division

It makes this easier!

Thus $y = (x+1)$ is the oblique asymptote of $f(x)$.

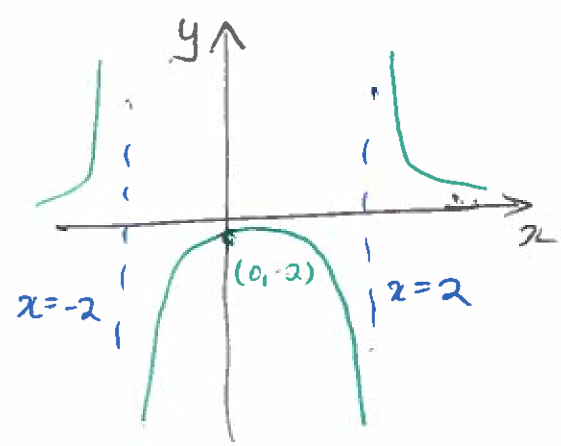
Curve Sketching (v2)

p204

1. Intercepts
2. Asymptotes — $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$
3. Derivatives — $f'(x)$ and $f''(x)$
4. Critical points — $f'(x)$ undef and $f'(x)=0$
5. Increasing/Decreasing
   Relative Extrema
6. Inflection Points — $f''(x)$ undef OR $f''(x)=0$
7. Concavity
8. Drawing!

<u>EX</u>: $f(x) = \dfrac{8}{x^2-4}$

- Make table.
- sketch graph.



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