Discussion Questions – Week #2b

Parker Glynn-Adey and Tyler Holden

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Please do the following questions as a group. Make sure that everyone in your group understands how each questions works. These questions are open ended and admit several approaches each. If you need help, please ask.

Question 1. *Suppose that $f : \mathbb{R}^k \to \mathbb{R}^\ell$ has the property that: if $S \subset \mathbb{R}^\ell$ is open then $f^{-1}(S) \subset \mathbb{R}^k$ is open. Show that f is continuous using the $\epsilon - \delta$ definition of continuous.*

Question 2. *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continous and $f(q) = g(q)$ for every rational $q \in \mathbb{Q}$. Show that $f(x) = g(x)$ for every $x \in \mathbb{R}$. (This shows that you can recover a continous function from its values of a dense subset. This is very useful.)*

Question 3. *Fix $n > 1$. Construct a function $f(x, y)$ such that:*

$$\lim_{(x,y)\to\mathbf{0}} f(x, y) = 0$$

for all paths of the form $(x, y) = (t, t^k)$ for $k < n$ but

$$\lim_{(x,y)\to\mathbf{0}} f(x, y) \neq 0$$

for the path $(x, y) = (t, t^n)$. (Try $n = 2$ first.)

Puzzle 1. *Let $I_0 = [0, 1]$ and define I_k to be the interval I_{k-1} obtained by removing the open middle third of each subinterval. For example,*

$$I_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], \qquad I_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Let $\mathcal{C} = \bigcap_{k=0}^{\infty} I_k$ be Cantor's (World Famous) Dust. Show that $\mathcal{C} \neq \emptyset$. Find as many points in $\mathcal{C}$ as you can.