Cops and Robbers with Many Variants

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Graphs and related definitions

Definition

A **simple graph** is an ordered pair $G = (V, E)$, where V is a set of vertices, and E is the set of edges.

Definition

A graph is connected if there are paths between any two of its vertices.

In this talk, we will only deal with finite and connected graphs.

Introduction

Definition

Cops and Robbers is a game played on graphs between an opposing set of cops and a single robber. The cops begin the game by moving to a set of vertices, with the robber then choosing a vertex to occupy. The players move from vertex-to-vertex along edges. The cops win by occupying the robber's vertex if possible.

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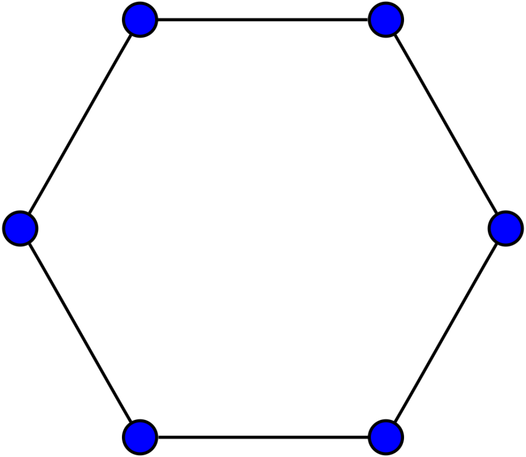
Cop-win and robber-win graph

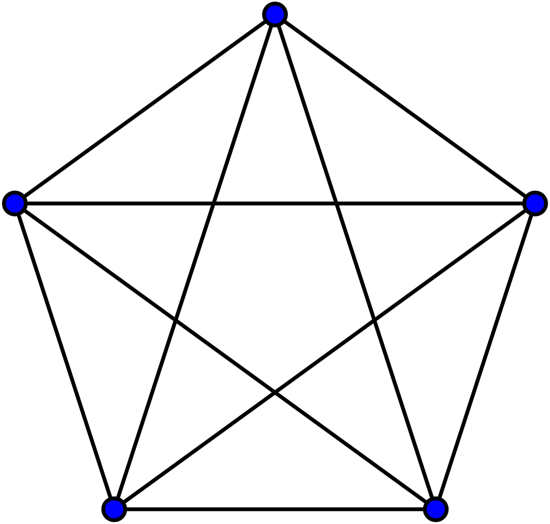
Definition

A **cop-win graph** is an undirected connected graph, where the cop can always win the game. Otherwise, the graph is a **robber-win graph**.

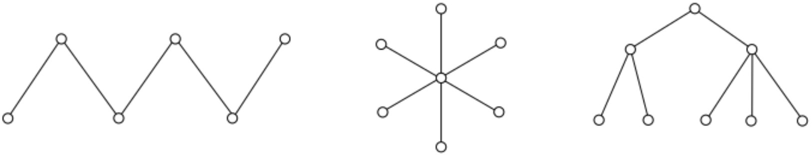
Example 1 (Cycle Graph)



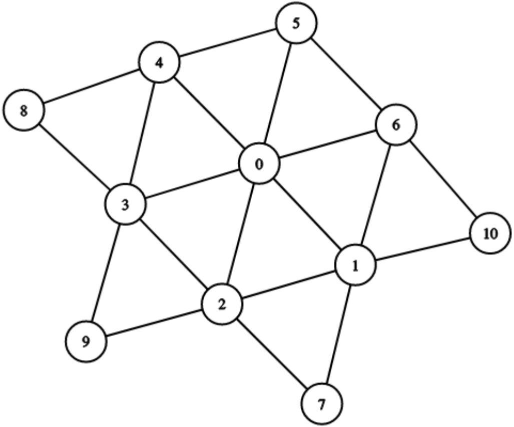
Example 2 (Complete Graph)



Example 3 (Trees)



But how about...

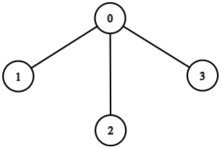


Motivated question

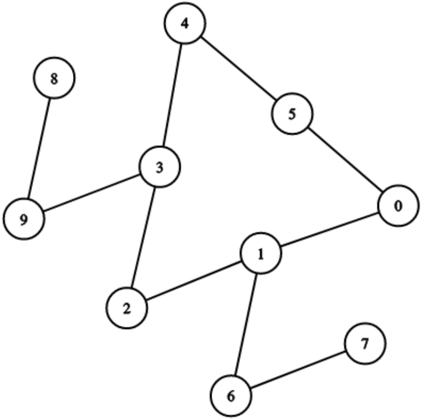
Can we quickly determine whether a graph is cop-win or robber-win?

Observation

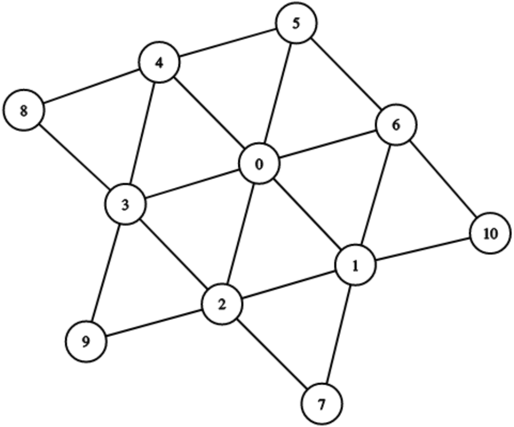
- We can see that there are some "dead-ends" in some graphs. We say that vertex $1, 2, 3$ is dominated by vertex 0 when the cop is at vertex 0.
- Removing a "dead-end" and its adjacent edges will not change who wins the game. (We can prove this by showing that adding a "dead-end" to a graph does not change the winner.)



Example



Example



Determine whether a graph is cop-win

Definition

The **open neighbourhood** of a vertex v in a graph G is the subgraph of G induced by all vertices adjacent to v, denoted by $N(v)$.

Definition

The **closed neighborhood** of a vertex v in a graph G is $N[v] = N(v) \cup \{v\}$.

Theorem

A finite graph is cop-win if and only if there is a linear ordering $v_0, \ldots, v_n$ of its vertices so that for each $i < n$ there is a $i < j \leq n$ such that $N[v_i] \subseteq N[v_j]$ (v_i is dominated by v_j). (Nowakowski and Winkler and Quilliot)

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Cop Number

Definition

The **cop number** of a graph G, written $c(G)$, is the minimum number of cops needed to win in G.

- All cop-win graph has cop-number 1.
- Cycle graph with more than 3 vertices?

Cop Number

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The **cop number** of a graph G, written $c(G)$, is the minimum number of cops needed to win in G.

- All cop-win graph has cop-number 1.
- Cycle graph with more than 3 vertices?
- Computing the cop-number of a graph is EXPTIME-complete (as complex as Chess played on an arbitrarily large, but finite, board). (William B Kinnersley, Anthony Bonato) [1]

Determining the cop number of a certain graph

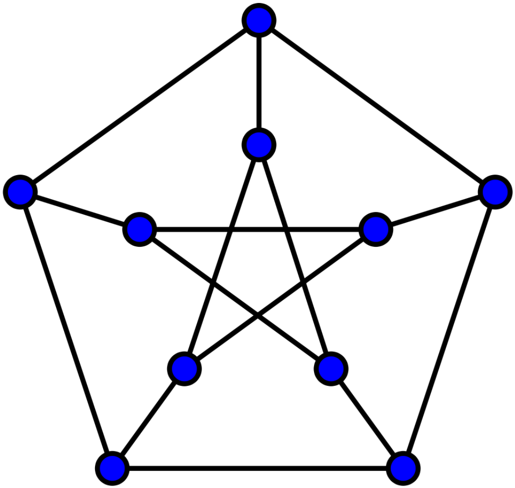
Theorem

Every planar graph has cop number at most three. (Aigner and Fromme, 1983) [2]

Definition

The **girth** of a graph is the length of the shortest cycle contained in the graph

Peterson Graph has Girth 5



Determining the cop number of a certain graph

Theorem

Let G be a graph with girth ≥ 5, its cop number is greater than or equal to the minimum degree of G.

Intuitively speaking, if every vertex of G has at least d adjacent edges but there are only $d - 1$ cops, then the robber can always escape through the unguarded edges. We need the restriction that girth ≥ 5 to eliminate the possibility of an individual cop guarding multiple edges at the same time.

Open Problems

- Mayniel's conjecture states that for a connected graph G, $c(G) = O(\sqrt{n})$, this is considered one of the deepest problems on the cop number. [3]
- Translation of the above statement: The largest possible cop number of an n-vertex graph is $\sqrt{n}$.
- Many more open problems appear when we look at the many variations of such game.

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Lazy Cops and Robbers

Despite having multiple cops, only one cop can move during each round. [4]
Note that this is only different from the regular cops and robbers if you have multiple cops.

Tipsy Cops and Robbers

- Tipsy move: a random move
- Non-tipsy move: the optimum move

A random movement chance, flavored as tipsiness, is included for both players, limiting their ability to execute winning strategies. In this iteration, both players move simultaneously in rounds and are not aware of the move the other player will make. [5]
Such variation has real-world applications such as how a white blood cell tracks down and eliminates bacteria in an organism.

Zombies and Survivors

The zombies, being of limited intelligence, have a very simple objective in each round—to move closer to a survivor. Therefore, each zombie must move along some shortest path on the graph.

Lemma

- *In any graph G, $c(G) \leq z(G)$, where $z(G)$ denotes the zombie number.*
- $z(T) = c(T) = 1$ *for any tree* T
- $z(C_n) = c(C_n) = 2$ *for* $n \geq 4$

Theorem

If a graph is zombie-win, then it is also cop-win. However, if a graph is cop-win, then it is not necessarily zombie-win. [6]

Some more variants

- Infinitely fast robber: the robber can traverse an arbitrarily long cop-free path. (William Kinnersley)
- Hyperopic cops and robber: imperfect information for cops, such that they cannot see the robber if the robber is within the n-neighborhood. A motivation for this comes from adversarial networks, where edges denote competition or rivalry between vertices. [7]

William B Kinnersley.
Cops and robbers is exptime-complete.
Journal of Combinatorial Theory, Series B, 111:201–220, 2015.

M. Aigner and M. Fromme.
A game of cops and robbers.
Discrete Applied Mathematics, 8(1):1–12, 1984.

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Meyniel's conjecture on the cop number: a survey, 2013.

Deepak Bal, Anthony Bonato, William B Kinnersley, and Paweł Prałat.
Lazy cops and robbers played on random graphs and graphs on surfaces.
2016.

Viktoriya Bardenova, Vincent Ciarcia, and Erik Insko.
Markov models for the tipsy cop and robber game on graphs, 2021.

S.L. Fitzpatrick, J. Howell, M.E. Messinger, and D.A. Pike.

A deterministic version of the game of zombies and survivors on graphs.
Discrete Applied Mathematics, 213:1–12, 2016.

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Hyperopic cops and robbers, 2017.