# Introduction to Combinatorial Games

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# Outline

# My goals

- I will show you a game, ways to modify it, and methods we can use to analyze them.
- I will share the terminology used to describe these games to help you find out more about combinatorial game theory.
- I will play the games we invent with you.
- I hope you will take one of these games with you today, to play or analyze or both, and then I hope you tell someone about it.

# Which games?

- Two people take turns removing 1 or 2 objects from a pile and *leaving* the rest, starting with **10**.
- The game ends when there are no possible moves. The last player to move wins.
- Who has a strategy to get the last move, the first or second player?
- Let's play!

# New games

- Two people take turns removing 1 or 2 objects from a pile and *leaving* the rest, starting with **10**.

## Attributes we will keep today

Two players taking turns; same moves allowed for both (we call this *impartial*); and the game eventually ends.

## Attributes we will change

The allowed moves 1 or 2, the starting amount **10**, and a secret third thing.

- One other common change is the last player to move **loses**. This is called *misere*. We won't use that rule today. They are interesting but harder to analyze - you can read more about them at `http://miseregames.org` by Thane Plambeck.

# Changing the pile

- The size of the pile changes during the game - making a move is like choosing a new **k** and trading places.

**Definition ($\mathcal{N}$ and $\mathcal{P}$ positions)**

We say **k** is an $\mathcal{N}$-position if the *next* player has a winning strategy, and a $\mathcal{P}$-position if the *previous* player does.

- How does the position value depend on **k**?
- Let's play!

**Theorem (Characterizing Theorem)**

- **k** *is in* $\mathcal{P}$ *if every* ($\forall$) *move from* **k** *lands in* $\mathcal{N}$*; and*
- **k** *is in* $\mathcal{N}$ *if some* ($\exists$) *move from* **k** *lands in* $\mathcal{P}$*.*

# Changing the moves

- A game with 1 or 2 only depends on the pile *mod* 3 - the remainder when dividing by 3.
- Something similar happens for 1 or 2 or 3.
- What happens with 1 or 3 or 4? What about 1 or 2 or 4?
- Let's play!
- With a finite set of moves, we will always get a repeating pattern. We just don't know what.
- We can use infinite move sets, like primes or *Subtract a Square*.

# Changing to piles

- Two people take turns removing 1 or 2 objects from a pile and *leaving or breaking* the rest, starting with **10**.
- Breaking means we separate the pile we take from into two piles.
- These games can have more than one pile, so we should clarify that we can *only break a pile we take from*.
- Let's play!

# Additional Changes

- The move set can change during the game.
- It can depend on the position, for example, any proper divisor of **k** gives a number theory twist.
- We can not allow the same move twice in a row. This means we can't use the same idea for a game tree, because moves depend on what happened, not just the current position (compare: Markov processes).
- *Octal games* let us specify how many piles we leave after taking a number. We played games allowing 0 or 1 (no breaking), then $0, 1$ or 2 (breaking allowed, not required).
- Let's play!

# Game Trees

- A *game tree* is a directed graph where vertices are game positions and edges are moves. Requiring the game to end means the graph has no cycles, so it is a tree!
- We could find all position values by climbing the tree.
- The difficulty is describing the patterns in the values. How high do we need to climb before we can see what is happening?
- We can generalize to other *combinatorial games on graphs*: moving tokens on a graph, or thinking of a graph as a pile and take smaller graphs from it, as in the *Caterpillar game*.
- Let's play!

# Modular ideas

- A game with 1 or 2 only depends on the pile mod 3 - that is, the remainder when dividing by 3.
- Something similar happens for 1 or 2 or 3.
- When we get a repeating pattern mod $n$, we're saying the remainders $\{0, 1, \ldots, n-1\}$ can be split into two sets, $\mathcal{N}$ and $\mathcal{P}$ satisfying the Characterizing Theorem.
- A game may be *eventually periodic*, so 0 mod $n$ may not be a $\mathcal{P}$ position, even though 0 must be.

# Breaking the binary

- In the game 2 and *leaving or breaking* the rest, every even position is $\mathcal{N}$, because we can subtract 2 and then break the rest into two equal piles.
- Some odd positions are also $\mathcal{N}$. But some are not. Let's play!
- If two positions are both $\mathcal{P}$, then so is their sum, using the "follow them" strategy. We could write $\mathcal{P} + \mathcal{P} = \mathcal{P}$.
- We also know $\mathcal{P} + \mathcal{N} = \mathcal{N}$, but what about $\mathcal{N} + \mathcal{N}$? Sometimes $\mathcal{N}$, sometimes $\mathcal{P}$.
- The *Sprague-Grundy value* encodes the value of all positions with smaller values it can move to. They are non-negative integers that you can find by "climbing the tree."
- The value is "the smallest non-negative integer that is **not** the value of a position it can move to." All $\mathcal{P}$ positions have value 0, so all $\mathcal{N}$ positions have positive value.

# Octal Games

- *Octal games* let us specify how many piles we leave after taking a number. We played taking games where we leave 1 pile (or 0), then breaking games where we can leave 0, 1 or 2.
- We might not allow 0 (i.e., you have to leave a non-empty pile), and we can use different rules for each subtraction.
- We can choose any subset of $0, 1, 2$ piles to be allowed, so there are 8 possible options *for each subtraction*. Hence, *octal* games.
- We know a lot about these games, but also very little.
- The database of known values of octal games is maintained by Achim Flammenkamp,
  `https://wwwhomes.uni-bielefeld.de/achim/octal.html`
- Let's play!

# Thanks

- I hope you liked these games! They're often called *subtraction* and taking-and-breaking games. They're easy to play anywhere, with anyone!
- Teach them to your younger relatives - they help think about logical implications and what someone else would do, then they can win against adults.
- Email me if you find a new game you think is fun!
- Check out Integers, the Electronic Journal of Combinatorial Number Theory, `http://math.colgate.edu/~integers/`. It has a games section and a special edition (2021B) last year about these topics.
- The books *Winning Ways for your Mathematical Plays* give a lot more games and analysis. The series *Games of No Chance* give surveys of what is known and new research results.