An introduction to the probabilistic method

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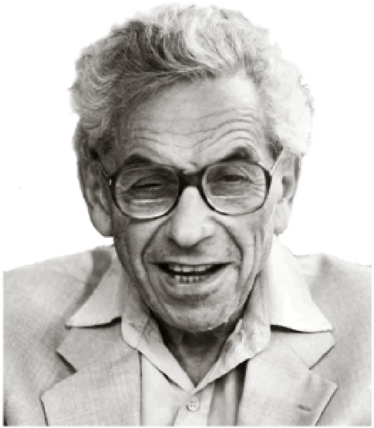
June 16, 2021

Erdős and the Probabilistic Method



Paul Erdős 1913 – 1996

Which of these is random?

Example #1: HHTHTHTHTHHTTTHTTTHTTHHHTHTTHHHT

Example #2: HTHTHTHTHTHTHTHTHTHTHTHTHTHTHTHT
This sequence is $x_n \equiv x_n + 1 \mod 2$.

Expectation

Definition

A random variable is a function $X : \Omega \to \mathbb{R}$. The expectation of a random variable is:

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \Pr(X = \omega)$$

Example

Parker and Lisa toss a coin.
If it comes up H, then Parker gives Lisa 500\$.
If it comes up T, then Lisa gives Parker 100\$.

Expectation

Example

Suppose that we toss a coin a hundred times, how many heads should appear?

Let X be the number of heads.

$$\mathbb{E}(X) = \sum_{k=0}^{100} k \Pr(X = k) = \sum_{k=0}^{100} k \binom{100}{k} \left(\frac{1}{2}\right)^{100-k} \left(\frac{1}{2}\right)^{k} = \frac{1}{2^{100}} \sum_{k=0}^{100} k \binom{100}{k} = \text{?!}$$

Linearity of Expectation

Lemma

The expectation $\mathbb{E}(\cdot)$

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \Pr(X = \omega)$$

is linear in X*. Formally,*

$$\mathbb{E}(\alpha X + \beta Y) = \alpha \mathbb{E}(X) + \beta \mathbb{E}(Y)$$

Linear of Expectation

Example

Suppose that we toss a coin a hundred times, how many heads should appear?

Let X be the number of heads. Write $X = X_1 + X_2 + \cdots + X_{100}$ where

$$X_k = \begin{cases} 1 & k'th\ coin\ heads \\ 0 & k'th\ coin\ tails \end{cases}$$

It follows from linearity of expectation that:

$$\mathbb{E}(X) = \sum_{i=1}^{100} \mathbb{E}(X_i) = \sum_{i=1}^{100} \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = 100 \cdot \frac{1}{2} = 50$$

The Probabilistic Method

Lemma (The Key Observation)

If $X \in \mathbb{N} \cup \{0\}$ *everywhere and* $\mathbb{E}(X) < 1$, *then* $X = 0$ *somewhere.*

Example (Silly?)

There exists a series of 100 coin tosses without 99 consecutive heads or tails.

We give a non-constructive proof! Let X be the number of occurences of H^{99} or T^{99}. We have $X = X_0 + X_1$ where $X_0 = 1$ if the first 99 tosses agree and $X_0 = 0$ otherwise. Similarly, $X_1 = 1$ if the last 99 tosses agree and $X_1 = 0$ otherwise.

$$\mathbb{E}(X) = \mathbb{E}(X_0) + \mathbb{E}(X_1) = \left(1 \cdot 2 \cdot \frac{1}{2^{99}} + 0\right) + \left(1 \cdot 2 \cdot \frac{1}{2^{99}} + 0\right) = \frac{1}{2^{97}} \ll 1$$

Thus, there is some event where $X = 0$. For example,

THTTHHTHHTHTTTTTTTHTHHHTHHTTTHTTHTTHHHTTHHTTHHHTHH
THTHHHHTHHHHTTTHHHHHHTHHHTHHHTHHHHHTHTHTTHTTHHHHHH

Ramsey Numbers

Definition

The (k, k)-Ramsey number $R(k, k)$ is the least n such that any edge-colouring of K_n by two colours Red and Blue contains a monochromatic K_k. For example, $R(3, 3) = 6$.

Ramsey Numbers

Theorem (Erdős, 1943)

$R(k,k)$ must grow exponentially in k.

Consider a random edge colouring of K_n.
There are $\binom{n}{2}$ edges, and we toss a Red/Blue coin for each edge.

Let S_k be a subset of k vertices.It defines a complete subgraph K_n.
We let $X(S_k) = 1$ if the K_k on S_k is monochrome and $X(S_k) = 0$ otherwise.

$$\mathbb{E}(X) = \Pr(S_k \ monochrome) = \left(\frac{1}{2}\right)^{\binom{k}{2}} + \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

Ramsey Numbers

And how many monochrome S_k can K_n have?
Let X be the total number of monochrome K_k in a random colouring.

$$\mathbb{E}(X) = \sum_{S_k \subset K_n} \mathbb{E}(X(S_k)) \leq |\{S_k \subset K_n\}| \Pr(K_k monochrome) \leq \binom{n}{k} 2^{1-\binom{k}{2}}$$

Lemma

If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ *then there is some colouring of* K_n *without any monochrome* K_k

Ramsey Numbers

We make the following estimates: $\binom{n}{k} < \frac{n^k}{k!}$ and $\frac{1}{2}k^2 < \binom{k}{2}$.
Suppose that $n = 2^{k/2}$ and $k > 3$ we get:

$$\begin{aligned} \mathbb{E}(X) &\leq \binom{n}{k} 2^{1-\binom{k}{2}} \\ &< \binom{n}{k} 2^{1-\frac{1}{2}k^2} \\ &< \frac{n^k}{k!} 2^{1-\frac{1}{2}k^2} \\ &< \frac{\left(2^{k/2}\right)^k}{k!} 2^{1-\frac{1}{2}k^2} = \frac{2}{k!} < 1 \end{aligned}$$

Theorem (Erdős, 1943)

$R(k,k)$ must grow faster than $2^{k/2}$.

Ramsey Numbers via Counting

How many colourings of K_n are there?
K_n has $\binom{n}{2}$ edges, and thus we get: $2^{\binom{n}{2}}$ colourings.

How many colourings of K_n have a monochromatic K_k?
Any S_k in K_n can be red or blue, and we let the remaining edges be coloured arbitrarily.
Thus, there are at most $2 \cdot \binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}}$ colourings with a monochrome K_k.

Thus, there will be colourings without monochrome K_k when:

$$2^{\binom{n}{2}} > 2 \cdot \binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \iff 1 > \binom{n}{k} 2^{1-\binom{k}{2}}$$

References

- Erdős, Paul. "Some remarks on the theory of graphs." *Bulletin of the American Mathematical Society* 53.4 (1947): 292-294.
- Alon, Noga, and Joel H. Spencer. *The probabilistic method.* John Wiley & Sons, 2016.

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