An Introduction to the Fractional Brownian Motion

Exposition & Insights

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The Classical Process

Motivation

Consider a finite collection of i.i.d. random variables $X_1, \ldots, X_n$ with $EX_i = 0$ and $EX_i^2 = 1$. Define a process $(Z_t)_{t\in[0,1]}$ by $Z\left(\frac{k}{n}\right) = X_k$ for $k \in \{1, \ldots, n\}$ and linearly interpolate on intervals of the form $\left[\frac{k}{n}, \frac{k+1}{n}\right)$.

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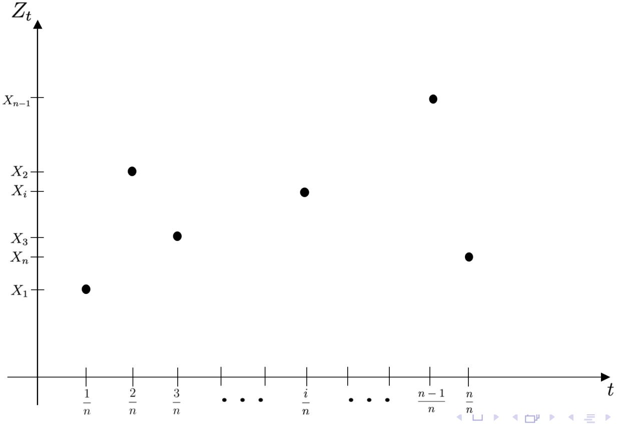

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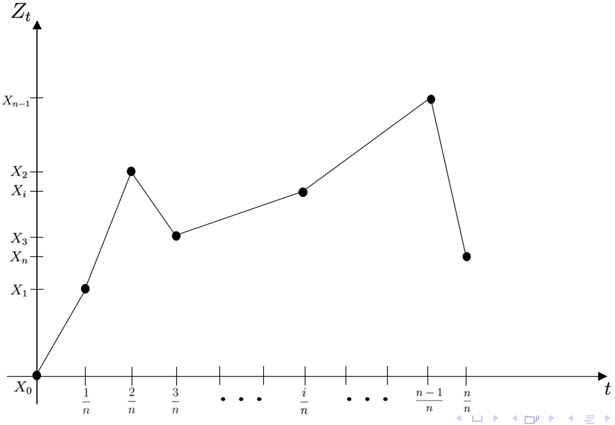

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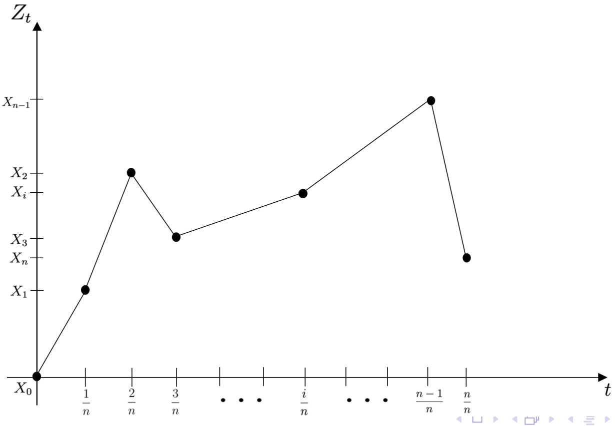

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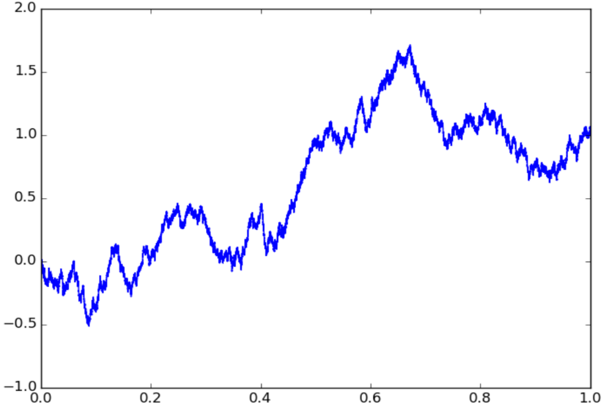

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Theorem: Characterization of Gaussian Processes

Gaussian processes are completely determined/characterized by their mean and covariance functions:

$$\begin{aligned} m : T &\to \mathbb{R} \qquad , \qquad & \Gamma : T \times T &\to \mathbb{R} \\ t &\mapsto E[X_t] & (s,t) &\mapsto \mathrm{Cov}(X_s, X_t). \end{aligned}$$

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Proof:

For a Gaussian process $(X_t)_{t\in T}$ with mean and covariance functions $m_{\mathbf{X}}, \Gamma_{\mathbf{X}}$ and any choice $t_1 < t_2 < \cdots < t_p$ we have $\mathbf{X} = (X_{t_1}, \ldots, X_{t_p})$, $t_i \in T$, to be an $\mathbb{R}^p$-valued random variable with a multivariate normal distribution and hence to have characteristic function (Fourier transform of distribution):

$$\varphi_{\mathbf{X}}(u) = E[e^{i\langle u,(X_{t_1},\ldots,X_{t_p})\rangle}] = \exp(i\langle u, \mu_{\mathbf{X}}\rangle - \frac{1}{2}\langle u, \mathbf{\Sigma_X} u\rangle),$$

where $\mu = (E[X_{t_1}], \ldots, E[X_{t_p}]) = (m(t_j))_{1\le j\le p}$ is the mean vector and $\mathbf{\Sigma}_{i,j} = (\Gamma(t_i, t_j))_{1\le i,j\le p}$ the covariance matrix.

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For a Gaussian process $(X_t)_{t\in T}$ with mean and covariance functions $m_{\mathbf{X}}, \Gamma_{\mathbf{X}}$ and any choice $t_1 < t_2 < \cdots < t_p$ we have $\mathbf{X} = (X_{t_1}, \ldots, X_{t_p})$, $t_i \in T$, to be an $\mathbb{R}^p$-valued random variable with a multivariate normal distribution and hence to have characteristic function (Fourier transform of its distribution):

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where $\mu_{\mathbf{X}} = (E[X_{t_1}], \ldots, E[X_{t_p}]) = (m_{\mathbf{X}}(t_j))_{1\le j\le p}$ is the mean vector and $(\mathbf{\Sigma}_{\mathbf{X}})_{i,j} = (\Gamma_{\mathbf{X}}(t_i, t_j))_{1\le i,j\le p}$ the covariance matrix. Now, considering another Gaussian process $(Y_t)_{t\in T}$ with the same mean and covariance functions as $(X_t)_{t\in T}$ i.e. $\Gamma_{\mathbf{Y}} = \Gamma_{\mathbf{X}}$, and $m_{\mathbf{X}} = m_{\mathbf{Y}}$, we have the characteristic function of its finite dimensional marginal, $(Y_{t_1}, \ldots, Y_{t_p})$, to be given identically as to that of above.

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Hence, we have an agreement in distribution of the finite dimensional marginals of $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ and hence an equivalence in law.

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Gaussian Processes

Definition: Centered Gaussian Process

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i.e. Gaussian processes sharing the same mean and cov. functions are equal in law.

Moral Conclusion:

Specifying Gaussian processes amounts to specifying a mean and covariance structure on the collection. Similarly as to how specifying a Gaussian r.v. requires only a mean and variance!

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Definition: Centered Gaussian Process

A **centered Gaussian process** is a stochastic process $(X_t)_{t \in T}$ such that any finite linear combination of the variables $X_t, t \in T$ is centered Gaussian. $(t \mapsto E[X_t] \equiv 0)$

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Definition: Brownian motion

A stochastic process $(B_t)_{t \geq 0}$ on $(\Omega, \mathscr{F}, P)$ is a standard **Brownian Motion** if it satisfies one of the following three equivalent assertions:

(i) $(B_t)_{t \geq 0}$ is a centered Gaussian process with covariance: $\text{Cov}(B_s, B_t) = \min(s,t) := s \wedge t$.

(ii) $B_0 = 0$ a.s. and for every $0 \leq s < t$ the random variable $B_t - B_s$ is independent of $\sigma(B_r : r \leq s)$ and $B_t - B_s \sim \mathcal{N}(0, t-s)$.

(iii) $B_0 = 0$ a.s. and for $0 = t_0 < t_1 < \cdots < t_n$ the r.v.'s $(B_{t_i} - B_{t_{i-1}})$ for $1 \leq i \leq n$ are indep. and $B_{t_i} - B_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$.

and as well:

(iv)$(B_t)_{t \geq 0}$ has surely continuous sample paths: $\forall \omega \in \Omega, t \mapsto B_t(\omega) \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$.

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Note:

Independence of increments are a key feature in the classical process!

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Proof:

Fix $0 \leq s < t$ and observe

$$E[B_s^{1/2} B_t^{1/2}] = \frac{1}{2}(t + s - |t-s|) = s = s \wedge t.$$

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Properties

Self-Similarity

For $\varepsilon > 0$ given and $H \in (0, 1)$, we have $(\varepsilon^{-H} B^H_{\varepsilon t})_{t \geq 0}$ is an fBm of Hurst index H.

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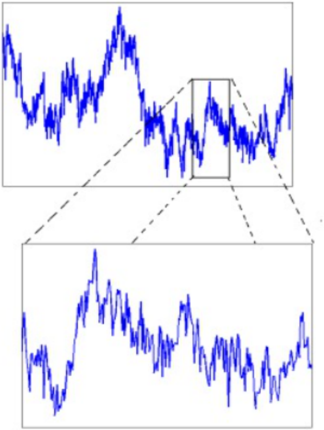
Illustration for $H = 1/2$:



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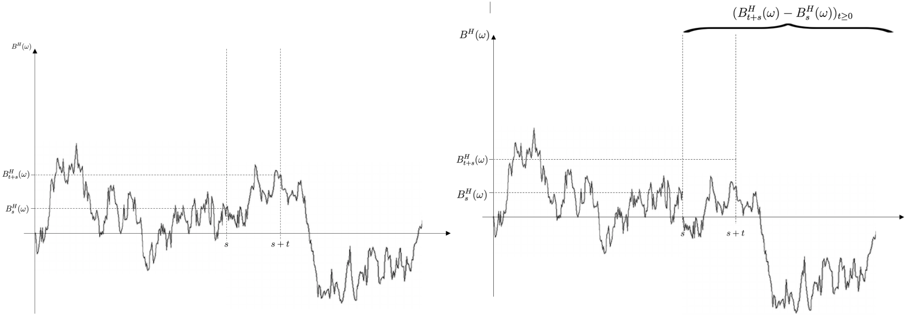

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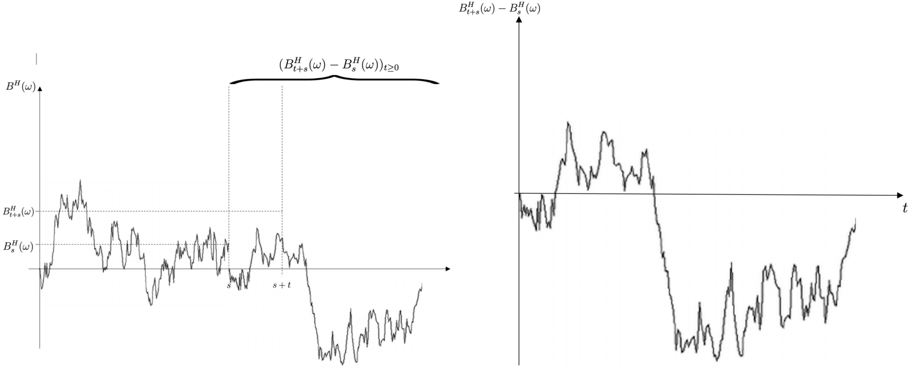

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An fBm $(B^H_t)_{t \geq 0}$ admits a continuous modification. That is we have some process $(X_t)_{t \geq 0}$ such that $t \mapsto X_t \in C[0, \infty)$ (surely) and for all $t \geq 0$, $P(B^H_t = X_t) = 1$.

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Theorem: Kolmogorov-Čhenstov Continuity Theorem

Assume that for a stochastic process $(X_t)_{t\geq 0}$ there exists $K > 0, p > 0, \beta > 0$ such that for all $s, t \geq 0$:

$$E[|X_t - X_s|^p] \leq K|t-s|^{1+\beta}.$$

Then the process has a continuous modification, i.e. a process $(\widetilde{X}_t)_{t\geq 0}$ such that $t \mapsto \widetilde{X}_t \in C[0,\infty)$ and for all $t \geq 0$ $P(X_t = \widetilde{X}_t) = 1$.

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Proof:

Simply observe:

$$E[(B_t^H - B_s^H)^2] = |t-s|^{2H},$$

and apply Kolmogorov-Čhenstov.

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Then how do we describe the dependence structure of fBm and how does such structure vary with the Hurst index?

Properties

Dependence of Increments

Disjoint increments of an fBm of Hurst index $H \in (0, 1]$ are *negatively* correlated for $H \in (0, 1/2)$ and *positively* correlated for $H \in (1/2, 1)$.

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Suppose that $0 \leq s_1 < t_1 < s_2 < t_2$ so as to ensure $[s_1, t_1] \cap [s_2, t_2] = \emptyset$. Then one can check the covariance of the increments to be given as:

$$E[(B_{t_1}^H - B_{s_1}^H)(B_{t_2}^H - B_{s_2}^H)] = \frac{1}{2}\left(|t_2 - s_1|^{2H} - |t_2 - t_1|^{2H} - \left(|s_2 - s_1|^{2H} - |s_2 - t_2|^{2H}\right)\right).$$

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Then considering the map $f(x) = x^{2H}$ and putting $a_1 = t_2 - s_1, a_2 = t_2 - t_1, b_1 = s_2 - s_1, b_2 = s_2 - t_1$ gives that $a_1 - a_2 = b_1 - b_2 = t_1 - s_1$ (note that $b_2 < a_2 < b_1 < a_1$) and allows the above to be expressed as follows:

$$E[(B^H_{t_1} - B^H_{s_1})(B^H_{t_2} - B^H_{s_2})] = \frac{1}{2}(f(a_1) - f(a_2) - (f(b_1) - f(b_2))).$$

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Disjoint increments of an fBm of Hurst index $H \in (0, 1]$ are *negatively* correlated for $H \in (0, 1/2)$ and *positively* correlated for $H \in (1/2, 1)$.

Proof:

Suppose that $0 \leq s_1 < t_1 < s_2 < t_2$ so as to ensure $[s_1, t_1] \cap [s_2, t_2] = \emptyset$. Then one can check the covariance of the increments to be given as:

$$E[(B_{t_1}^H - B_{s_1}^H)(B_{t_2}^H - B_{s_2}^H)] = \frac{1}{2}\left(|t_2 - s_1|^{2H} - |t_2 - t_1|^{2H} - \left(|s_2 - s_1|^{2H} - |s_2 - t_2|^{2H}\right)\right).$$

Then considering the map $f(x) = x^{2H}$ and putting $a_1 = t_2 - s_1, a_2 = t_2 - t_1, b_1 = s_2 - s_1, b_2 = s_2 - t_1$ gives that $a_1 - a_2 = b_1 - b_2 = t_1 - s_1$ (note that $b_2 < a_2 < b_1 < a_1$) and allows the above to be expressed as follows:

$$E[(B_{t_1}^H - B_{s_1}^H)(B_{t_2}^H - B_{s_2}^H)] = \frac{1}{2}(f(a_1) - f(a_2) - (f(b_1) - f(b_2))).$$

Now as $f'' < 0$ for $H \in (0, 1/2)$ we have for such H that $E[(B_{t_1}^H - B_{s_1}^H)(B_{t_2}^H - B_{s_2}^H)] < 0$. And since $f'' > 0$ for $H \in (1/2, 1)$ we have for such H that $E[(B_{t_1}^H - B_{s_1}^H)(B_{t_2}^H - B_{s_2}^H)] > 0$.

□

Sample Path Regularity w.r.t H

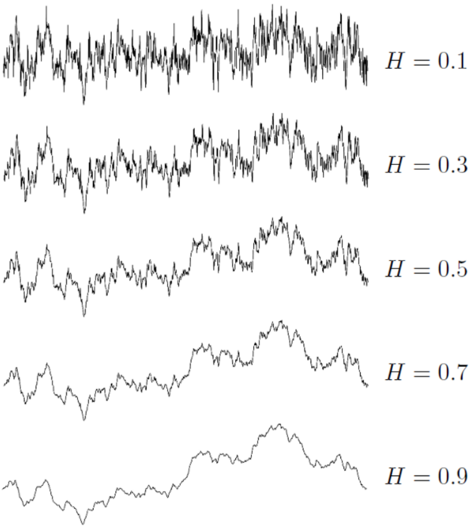

FIGURE 1. Paths of fBm for different values of H.

Limiting Case

Gaussian Beam; H = 1

One can check that for $H = 1$ we have $(B_t^H)_{t\geq 0} \overset{d}{=} (t\xi)_{t\geq 0}$, for $\xi \sim \mathcal{N}(0,1)$.

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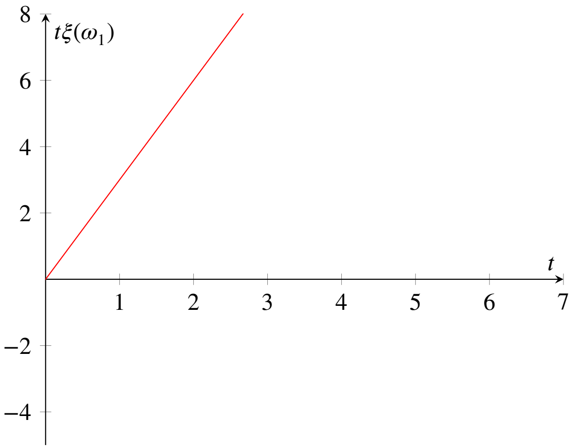

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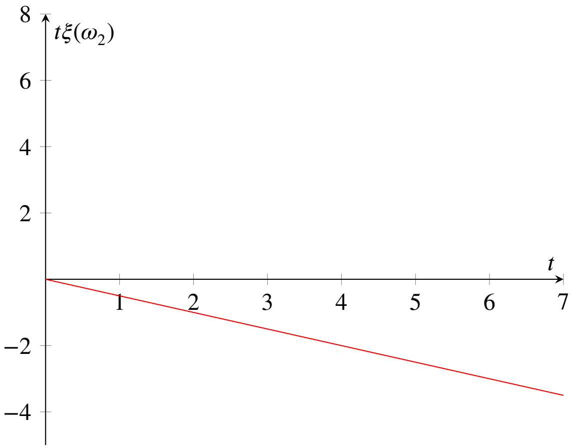

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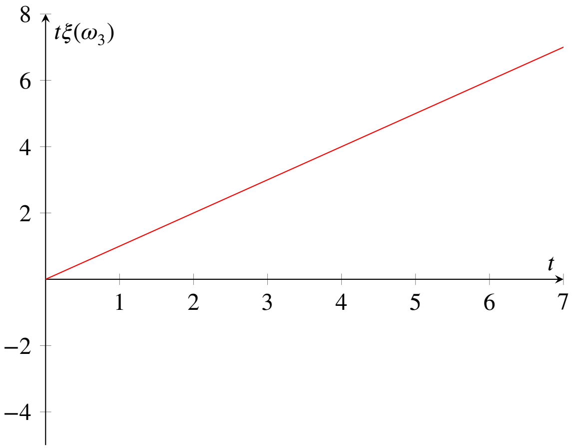

References

[1] J. Bertoine, Stochastics MAT901, University of Zurich^{UZH}, Spring 2018.

[2] Z. Copur, *Handbook of Research on Behavioural Finance and Investment Strategies: Decision Making in the Financial Industry*, Hacettepe University, 2015

[3] R. Durrett, *Probability: Theory and Examples*, Cambridge Series in Statistical and Probabilistic Mathematics (49), Cambridge University Press, 5th edition, 2019.

[4] J. Jacod, P. Protter, *Probability Essentials*, Universitext, Springer-Verlag, 2nd edition, 2004.

[5] S. Lalley, *Brownian Motion*, Statistics 385: Brownian Motion and Stochastic Calculus, University of Chicago, 2016.

[6] J.-F. Le Gall, *Brownian Motion, Martingales and Stochastic Calculus*, Graduate Texts in Mathematics (274), Springer-Verlag, 2016.

[7] P. Mörters, Y. Peres, *Brownian Motion*, Cambridge Series in Statistical and Probabilistic Mathematics (30), Cambridge University Press, 1st edition, 2010.

[8] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series (30), Princeton University Press, 2016.

[9] T-C. Wang, *Solutions to Exercises on Le Gall's Book: Brownian Motion, Martingales, and Stochastic Calculus*, Department of Applied Mathematics National Chiao Tung University, 2021.

[10] I. Nourdin, *Selected Aspects of Fractional Brownian Motion*, Bocconi & Springer Series - Mathematics, Statistics, Finance and Economics (4), Bocconi University Press, 1st edition, Springer-Verlag, 2012.

[11] Georgiy Shevchenko, *Fractional Brownian Motion in a Nutshell*, 8th Jun. 2014.