Enumerative Geometry: Past, Present, and Future

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Joint with P. Breiding (Osnäbruck), J. Lindberg (UT Austin), and L. Sommer (Leipzig)
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Given five conic sections in the plane, how many conic sections are tangent to all five?

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Euclid, 300 BCE

There is one line passing through two points in the plane.

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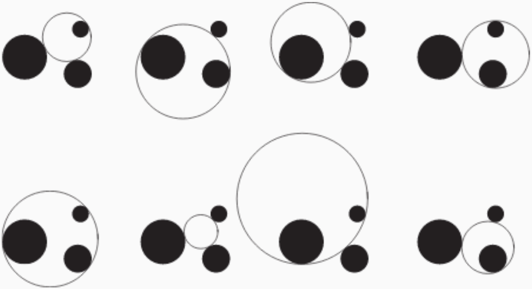
Given three circles in the plane, how many circles are tangent to all three?

Apollonius, 200 BCE

Given three circles in the plane, there are eight circles tangent to all three.



https://mathworld.wolfram.com/ApolloniusProblem.html

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Chasles/de Jonriques, 1860

Given five conic sections in the plane, there are 3264 conic sections tangent to all five.

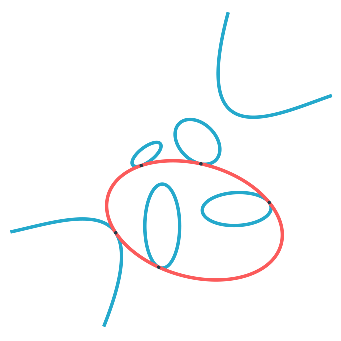

Figure 1 of https://www.ams.org/journals/notices/202001/rnoti-p30.pdf

A circle in the plane is determined by three parameters: the two coordinates for its center, and its radius.

A circle with center (a, b) and radius r has equation

$$(x-a)^2 + (y-b)^2 - r^2 = 0.$$

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Answers to Enumerative Problems

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- The condition "the circle C is tangent to a given circle C'" is a 1-dimensional constraint on the 3-dimensional space of circles. In other words, given a circle C', there is a 2-dimensional family of circles C that are tangent to C'.
- Having a circle C being simultaneously tangent to three circles C_1, C_2, C_3 is equivalent to these three 2-dimensional subspaces of the 3-dimensional space of circles intersecting.
- Three surfaces (2-dimensional subspaces) of a 3-dimensional space meet in a 0-dimensional (finite) set.

Question (Apollonius, 200 BCE)

Given *three* circles in the plane, how many circles are tangent to all three?

Question (Breiding, 2018)

Given three conics in the plane, how many circles are tangent to all three?

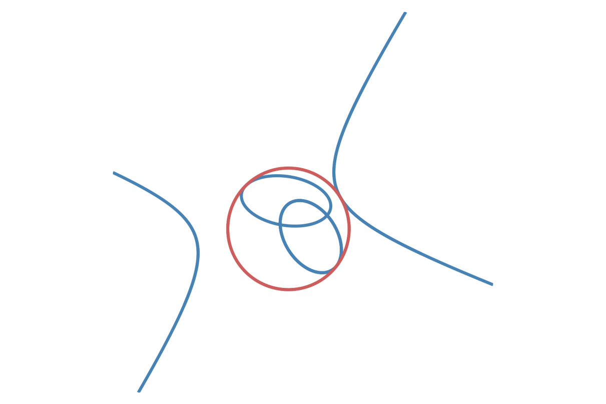

Figure 1 of *arXiv: 2211.06876*.

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Emiris and Tzoumas, 2005

Given three *ellipses* in the plane, there are *at most* 184 circles tangent to all three.

Circles Tangent to Three Conics

Question (Breiding, 2018)

Given three conics in the plane, how many circles are tangent to all three?

Emiris and Tzoumas, 2005

Given three *ellipses* in the plane, there are *at most* 184 circles tangent to all three.

Breiding-Lindberg-O.-Sommer, 2022

Given three *general conics* in the plane, there are *exactly* 184 circles tangent to all three.

A White Lie

The following theorems were proved over $\mathbb{C}$ and boil down to counting the number of roots of (multivariate) polynomials.

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Question

What versions of the above theorems hold over $\mathbb{R}$?

Questions in real enumerative geometry are of intrinsic interest, but are also linked to questions in applied mathematics.

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What versions of the above theorems hold over $\mathbb{R}$?

Questions in real enumerative geometry are of intrinsic interest, but are also linked to questions in applied mathematics.

It makes sense to tell a robot to move 2.1m forward, less so to move $1.3 + 2\sqrt{-1}$m forward.

Real Enumerative Problems

Real enumerative geometry asks questions about **real** geometric objects of the form “how many” and expects answers in $\mathbb{N}$.

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Question (*a la* Euclid, 300 BCE)

Given two real points in the real plane, how many real lines pass through both points?

Question (*a la* Apollonius, 200 BCE)

Given three real circles in the real plane, how many real circles are tangent to all three?

Question (*a la* Steiner, 1850s)

Given five real conic sections in the plane, how many real conic sections are tangent to all five?

What can be done over the real numbers?

Over $\mathbb{R}$, we lose "invariance of number". We no longer have well-defined solution counts. Solutions appear in a range, just as a quadratic polynomial can have 0 or 2 roots.

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Euclid, 300 BCE

Given two real points in the real plane, there is exactly one real line passing through both.

Pedoe, 1970

Given three real cicles in the real plane, there can be $\{0, 1, 2, 3, 4, 5, 6, 8\}$ real circles tangent to all three.

Let us consider Steiner's problem over $\mathbb{R}$.

Ronga, et. al., 1997

There exists a configuration of five real conics in the real plane with 3264 real conics tangent to all five.

Question

What numbers of conics tangent to five conics can be attained? Can you give a list as in the case of the real Apollonius' problem?

Question (Breiding, 2018)

Given three conics in the plane, how many circles are tangent to all three? How many of the tangent circles can be real?

Real Circles Tangent to Three Conics

Question (Breiding, 2018)

Given three conics in the plane, how many circles are tangent to all three? How many of the tangent circles can be real?

Breiding-Lindberg-O.-Sommer

There exists a configuration of three real conics with 136 real tangent circles.

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For every $k \in \{0, 2, 4, \ldots, 136\}$, there exists a configuration of three real conics with k real tangent circles.

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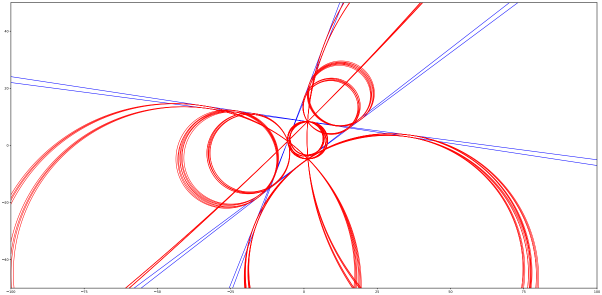
For every $k \in \{0, 2, 4, \ldots, 136\}$, there exists a configuration of three real conics with k real tangent circles.

Conjecture

There are at most 136 real tangent circles.


40
20
0
−20
−40
−100
−75
−50
−25
0
25
50
75
100

A conic is determined by its 6 coefficients

$$a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$$

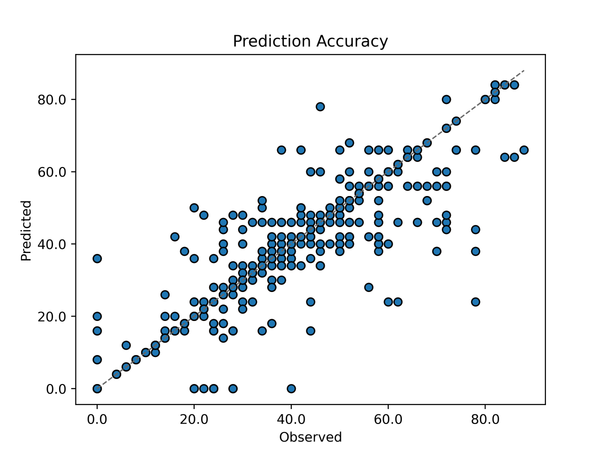
so a configuration of three conics is determined by $3 \times 6 = 18$ coefficients.

Question

Given three conics (as defined by a vector of length 18), how many *real* circles are tangent to all three?

Classifying Solutions

Machine learning provides us a powerful tool for understanding the real solution structure of this tangency problem.



Referee's Report

"I am skeptical on the general feasibility of the machine learning approach for classification problems in (real) algebraic geometry, but this is rather a philosophical comment and it is clear that the question by itself is interesting ..."

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How else can we use machine learning in (real) geometry?

Definition (Discriminant)

The discriminant is a subspace dimension $n-1$ in the n-dimensional space of coefficients that tells us when a polynomial has a root of multiplicity.

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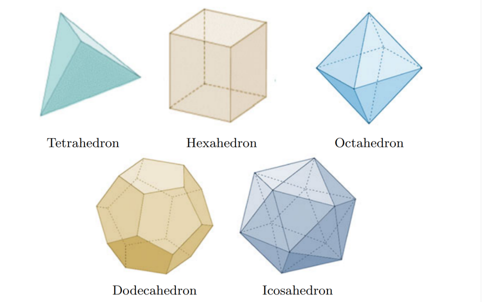
In work forthcoming work with Anna Seigal, we show that we can "learn" the quadratic discriminant $b^2 - 4ac$ to high accuracy:

$$b^2 - 4.00142606023ac.$$

In the **past**, enumerative geometry has been the source of many beautiful questions in mathematics.

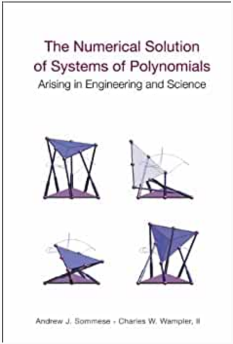


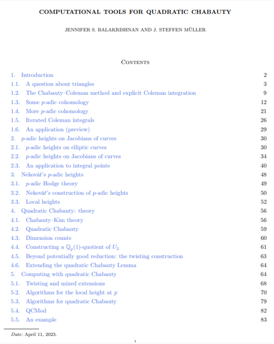
(Left) The Platonic solids. (Right) Clay tablet IM 67118 with computations of Pythagorean triples, Iraq museum.

(Left) https://link.springer.com/chapter/10.1007/978-981-16-6108-2_8
(Right) https://en.wikipedia.org/wiki/IM_67118

Geometry, Computation, and the Present

In the **present**, we have seen how these classical questions are closely tied to problems that arise both within mathematics and in applications.




COMPUTATIONAL TOOLS FOR QUADRATIC CHABAUTY

JENNIFER S. BALAKRISHNAN AND J. STEFFEN MÜLLER

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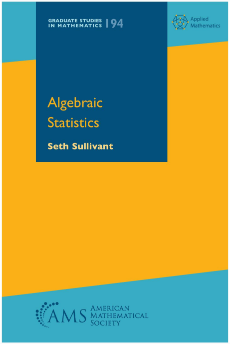
Date: April 11, 2023.



(L) https://www.worldscientific.com/worldscibooks/10.1142/5763#t=aboutBook
(C) http://math.bu.edu/people/jbala/2020BalakrishnanMuellerNotes.pdf
(R) https://bookstore.ams.org/gsm-194/

Today, we are also very lucky to have a wide range of computational tools to apply to our mathematical problems.

- Computational commutative algebra and algebraic geometry in *Macaulay2*
- Computational number theory in *SageMath*
- Numerical algebraic geometry in *HomotopyContinuation.jl*
- Computational group theory in *GAP*
- And many more...

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There are many more questions and connections for **you** to go out there and explore.

Thank You. Questions?