Periods: from pendula to the present

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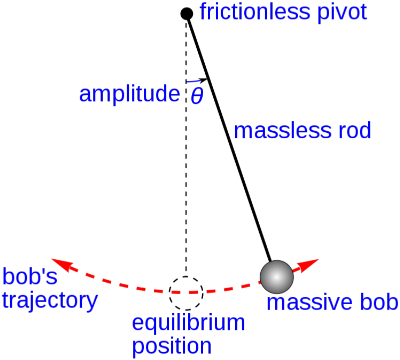
Based on the survey article “Periods” by Kontsevich and Zagier

The pendulum



length ℓ mass m

$$\text{kinetic} = \frac{1}{2} m\ell^2 \left(\frac{d\theta}{dt}\right)^2$$

$$\text{potential} = -mg\ell\cos\theta$$

Dropped from rest at $\theta = \frac{\pi}{6}$

$$\frac{1}{2} m\ell^2 \left(\frac{d\theta}{dt}\right)^2 - mg\ell\cos\theta = -\frac{\sqrt{3}mg\ell}{2}$$

so that

$$dt = \frac{d\theta}{\sqrt{\frac{2g}{\ell}\left(\cos\theta - \frac{\sqrt{3}}{2}\right)}}$$

Period of oscillation:

$$\begin{aligned} T &= 2\int_{-\pi/6}^{\pi/6} \frac{d\theta}{\sqrt{\frac{2g}{\ell}(\cos\theta - \frac{\sqrt{3}}{2})}} \\ &= 2\sqrt{\frac{\ell}{g}} \int_0^1 \frac{dx}{\sqrt{(1-x^2/16)(1-x^2)}} \end{aligned}$$

Periods

$$\mathbb{Q}[x_1, \ldots, x_n] := \{\text{polynomials in } x_1, \ldots, x_n \text{ with coefficients in } \mathbb{Q}\}$$

$$\frac{5}{4}x + x^2 y \in \mathbb{Q}[x, y] \qquad \pi x \notin \mathbb{Q}[x] \qquad \sin(x) \notin \mathbb{Q}[x]$$

Definition (Kontsevich–Zagier)

A real number is a **period** if it can be written as an absolutely convergent integral

$$I = \int_D \frac{g}{h} dx_1 \cdots dx_n$$

where

- $g, h \in \mathbb{Q}[x_1, \ldots, x_n]$
- $D \subset \mathbb{R}^n$ defined by $f_j(x_1, \ldots, x_n) \geq 0$, for $f_1, f_2, \ldots, \in \mathbb{Q}[x_1, \ldots, x_n]$

A complex number is a period if its real and imaginary parts are periods.

Equivalent definition: also allow "algebraic" functions like $\sqrt{-}$

Examples of periods

- Every $a \in \mathbb{Q}$ is a period:

$$a = \int_0^a dx = \int_{0 \le x \le a} 1\, dx$$

- π is a period:

$$\pi = \text{area of unit disk} = \int_{x^2+y^2 \le 1} dx\, dy$$

- If $a \in \mathbb{Q}_{>0}$ then $\sqrt{a}$ is a period:

$$\sqrt{a} = \int_0^{\sqrt{a}} dx = \tfrac{1}{2} \int_{-\sqrt{a}}^{\sqrt{a}} dx = \int_{0 \le x^2 \le a} \tfrac{1}{2}\, dx$$

- If $0 \neq a \in \mathbb{Q}$ then $\log(a)$ is a period:

$$\log a = \log(a) - \log(1) = \int_{1 \le x \le a} \frac{1}{x}\, dx$$

Periods vs. non-periods

$$\mathscr{P} := \{\text{periods}\} \subset \mathbb{C}$$

- $\mathbb{Q}$ countable $\implies$ $\mathbb{Q}[x_1, \ldots, x_n]$ countable $\implies$ $\mathscr{P}$ countable
- $\mathbb{R}, \mathbb{C}$ uncountable $\implies$ "most" numbers are *not* periods
- Hierarchy:

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \{\text{roots of } f \mid f \in \mathbb{Q}[x]\} \subsetneq \mathscr{P} \subsetneq \{\text{computable \#s}\} \subsetneq \mathbb{C}$$

- Not a period: Chaitin's constant := probability that a random algorithm halts
- No proven "natural" examples of non-periods
- Conjecture: $e \notin \mathscr{P}$ — but $e = \sum_{n=1}^{\infty} 1/n!$, so it's computable

Another example

$$
\begin{aligned}
\mathscr{P} \quad &\ni \quad \int\int_{0<x<y<1} \frac{dx\,dy}{(1-x)y} \\
&= \quad \int_0^1 \left(\int_0^y \frac{dx}{1-x} \right) \frac{dy}{y} \\
&= \quad \int_0^1 -\log(1-y) \frac{dy}{y} \\
&= \quad \int_0^1 \sum_{n\geq 1} \frac{y^n}{n} \frac{dy}{y} \\
&= \quad \sum_{n\geq 1} \frac{y^n}{n^2} \Bigg|_{y=0}^{y=1} \\
&= \quad \sum_{n\geq 1} \frac{1}{n^2} \\
&= \quad \frac{\pi^2}{6} \qquad\qquad \text{(Euler 1734)}
\end{aligned}
$$

Transcendence of zeta values

More generally, for $1 < k \in \mathbb{Z}$:

$$\zeta(k) := \sum_{n\geq 1} \frac{1}{n^k} = \int_{0<x_1<x_2<\cdots<x_n<1} \frac{dx_1\, dx_2\, \cdots\, dx_n}{(x_1-1)x_2x_3\cdots x_n} \in \mathscr{P}$$

Theorem (Euler 1735): $\zeta(2m) = (-1)^{m+1}\frac{B_{2m}(2\pi)^{2m}}{2(2m)!} \in \mathbb{Q}\pi^{2m}$

Open Question: Is $\zeta(2m+1) \in \mathbb{Q}[\pi]$?

Conjecture: $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

Theorem (Apéry 1978): $\zeta(3) \notin \mathbb{Q}$

Theorem ((Ball–)Rivoal 2000): Infinitely many $\zeta(3), \zeta(5), \zeta(7), \ldots \notin \mathbb{Q}$

Theorem (Zudilin 2000): At least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11) \notin \mathbb{Q}$

How to tell if two periods are equal?

"Obvious" tricks:

- Additivity:

$$\int_{D_1 \sqcup D_2} = \int_{D_1} + \int_{D_2} \qquad \int (f+g) = \int f + \int g$$

- Change of variables formula
- Fubini's theorem
- Fundamental theorem of calculus

Conjecture (Version of the Grothendieck Period Conjecture)

If two periods are equal, one can prove they are equal using the identities above, and nothing more.

Known to imply previous conjecture on independence of $\pi, \zeta(3), \zeta(5), \ldots$

Example

$$\zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Can be proven as follows:

$$\begin{aligned}
3\sum_{n>0}\frac{1}{n^2} &= \int_0^1\int_0^1 \frac{1}{1-xy}\frac{dx\,dy}{\sqrt{xy}} && \text{(series expansion)} \\
&= 4\int_{\xi,\eta>0,\xi\eta\le 1} \frac{d\xi\,d\eta}{(1+\xi^2)(1+\eta^2)} && \left(x=\xi^2\tfrac{1+\eta^2}{1+\xi^2}, y=\eta^2\tfrac{1+\xi^2}{1+\eta^2}\right) \\
&= 2\int_0^\infty \frac{d\xi}{1+\xi^2}\int_0^\infty \frac{d\eta}{1+\eta^2} && \text{(Fubini)} \\
&= 2\left(\frac{\pi}{2}\right)^2 && \text{(FTC)}
\end{aligned}$$

Toolbox: algebraic geometry

Definition

An **(affine) algebraic variety** X (over the rational numbers) is the set of solutions* of a system of polynomial equations $f_1 = \cdots = f_m = 0$ where $f_1, \ldots, f_m \in \mathbb{Q}[x_1, \ldots, x_n]$.

*consider solutions taking values in any field $k \supseteq \mathbb{Q}$, e.g.

$$X(\mathbb{C}) := \{(x_1, \ldots, x_n) \in \mathbb{C}^n \,|\, f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0\}$$

Example

Let X defined by $xy - 1 = 0$ in the plane.

$$X(\mathbb{C}) = \left\{(x, y) \in \mathbb{C}^2 \,\middle|\, xy = 1\right\} = \{(x, 1/x) \,|\, 0 \neq x \in \mathbb{C}\} \cong \mathbb{C} \setminus \{0\}$$

More generally, for any $k \supset \mathbb{Q}$, we have $X(k) \cong k^\times := k \setminus \{0\}$

$$X = \mathbb{G}_m \qquad \text{“the multiplicative group”}$$

Regular functions

Definition

If X is an algebraic variety, a **regular function on** X is a function of the form $f|_X$ where $f \in \mathbb{Q}[x_1, \ldots, x_n]$.

Example

Consider $X = \mathbb{G}_m = \{xy = 1\}$. Then

$$(\tilde{x} : (x, y) \mapsto x) \in \mathcal{O}(X) \qquad (\tilde{y} := (x, y) \mapsto y) \in \mathcal{O}(X)$$

and we have

$$\tilde{y} = \tilde{x}^{-1}$$

so that

$$\mathcal{O}(X) \cong \mathbb{Q}[\tilde{x}, \tilde{x}^{-1}] = \{a_{-m}\tilde{x}^{-m} + \cdots + a_{-1}\tilde{x}^{-1} + a_0 + \cdots + a_l\tilde{x}^l\}$$

Generalizing vector calculus: line integrals

A **differential 1-form** $\alpha \in \Omega^1(X)$ is a formal expression

$$\alpha := g_1\, dx_1 + \cdots + g_n\, dx_n$$

where $g_j \in \mathcal{O}(X)$, e.g. if $f \in \mathcal{O}(X)$, then

$$df := \sum_j \frac{\partial f}{\partial x_j} dx_j \in \Omega^1(X)$$

A **path in** X is a differentiable map $\gamma : [a, b] \to X(\mathbb{C})$

$$\begin{aligned}\int_\gamma \alpha &:= \int_a^b \alpha(\gamma(t)) \\ &= \sum_j \int_a^b g_j(\gamma(t)) d(x_j(\gamma(t))) \\ &= \sum_j \int_a^b g_j(\gamma(t)) \frac{d(x_j(\gamma(t)))}{dt}\, dt \in \mathbb{C}\end{aligned}$$

Example of $\mathbb{G}_m$

$$\mathbb{G}_m := \{xy = 1\} \qquad \mathbb{G}_m(\mathbb{C}) \cong \mathbb{C}^\times \qquad \mathcal{O}(\mathbb{G}_m) \cong \mathbb{Q}[x, x^{-1}]$$

One-forms:

$$\alpha = (a_{-m}x^{-m} + \cdots + a_{-1}x^{-1} + a_0 + \cdots + a_n x^n)\, dx$$

Example

γ the straight line from 1 to $1 \neq a \in \mathbb{Q}$ in $\mathbb{G}_m(\mathbb{C})$, i.e. $\gamma(t) = t$, $t \in [1, a]$

$$\int_\gamma \frac{dx}{x} = \int_1^a \frac{dt}{t} = \log a \in \mathscr{P}$$

Example

γ the unit circle in $\mathbb{G}_m(\mathbb{C})$, i.e. $\gamma(t) = e^{it}$ with $t \in [0, 2\pi]$

$$\int_\gamma \frac{dx}{x} = \int_0^{2\pi} \frac{d(e^{it})}{e^{it}} = \int_0^{2\pi} \frac{ie^{it}\,dt}{e^{it}} = \int_0^{2\pi} i\,dt = 2\pi i \in \mathscr{P}$$

Path independence

Theorem (from Vector Calculus)

Let V be a vector field on $U \subset \mathbb{R}^3$.

- *If $V = grad(F)$, then $\int_\gamma V \cdot ds = F(\gamma(1)) - F(\gamma(0))$*
- *If $curl(V) = 0$, then $\int_\gamma V \cdot ds$ is invariant under deformation of γ*

Generalization: if

- $Y \subset X$ algebraic subvariety
- $\alpha \in \Omega^1(X)$ such that $\alpha|_Y = 0$
- γ a path in $X(\mathbb{C})$ with endpoints in $Y(\mathbb{C})$

then

$$\alpha = df \implies \int_\gamma df = f(\gamma(1)) - f(\gamma(0))$$

$$d\alpha := \sum_{i,j} \left(\frac{\partial g_j}{\partial x_i} - \frac{\partial g_i}{\partial x_j} \right) dx_i \wedge dx_j = 0 \implies \int_\gamma \alpha \quad \begin{array}{c} \text{is invariant under} \\ \text{deformations of } \gamma \\ \text{with endpoints in } Y(\mathbb{C}) \end{array}$$

Higher-dimensional integration

More generally: differential k-forms

$$\alpha := \sum g_{j_1,\dots,j_k} dx_{j_1} \wedge \cdots \wedge dx_{j_k} \in \Omega^k(X)$$

subject to $dx_j \wedge dx_l = -dx_l \wedge dx_j$.

Exterior derivative (generalizing grad, curl, div):

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

$$d\alpha = \sum \frac{dg_{j_1,\dots,j_k}}{dx_l} dx_l \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}$$

Given $\Psi : [0,1]^k \to X(\mathbb{C})$ define

$$\int_\Psi \alpha \in \mathbb{C},$$

FTC, grad/Stokes/div theorems generalize to **Stokes' theorem**:

$$\int_\Psi d\alpha = \int_{\text{boundary of } \Psi} \alpha$$

(Co)homology: "list" all integrals, modulo $\int_{\partial\Psi} \alpha = \int_{\Psi} d\alpha$

Definition ("integrands")

The **de Rham cohomology of** (X, Y) is the $\mathbb{Q}$-vector space

$$H^k_{dR}(X,Y) := \frac{\{\alpha \in \Omega^k(X) \,|\, \alpha|_Y = 0 \text{ and } d\alpha = 0\}}{\{d\lambda \,|\, \lambda \in \Omega^{k-1}(X) \text{ and } \lambda|_Y = 0\}}$$

Definition ("domains of integration")

The **Betti homology of** (X, Y) is the $\mathbb{Q}$-vector space

$$H^B_k(X,Y) := \frac{\mathbb{Q}\cdot\{k\text{-dim param. subsets of } X(\mathbb{C}) \text{ with boundary in } Y(\mathbb{C})\}}{\mathbb{Q}\cdot\{\text{boundaries of } (k+1)\text{-dim.}\} + \mathbb{Q}\cdot\{\text{subsets of } Y(\mathbb{C})\}}$$

Then have well-defined $\mathbb{Q}$-bilinear map

$$H^k_{dR}(X,Y) \times H^B_k(X,Y) \to \mathbb{C} \qquad (\alpha, \sum_j a_j \Psi_j) \mapsto \sum_j a_j \int_{\Psi_j} \alpha$$

Theorem: every number obtained in this way is a period.

Grothendieck's algebraic de Rham theorem

$$H^k_{dR}(X,Y) \times H_k^B(X,Y) \to \mathscr{P} \qquad \left(\alpha, \sum_j a_j \Psi_j\right) \mapsto \sum_j a_j \int_{\Psi_j} \alpha$$

Theorem (Grothendieck 1966, building on work of Cartan, de Rham, Hironaka, Poincaré, Serre, ...)

For all (X,Y) and k, the $\mathbb{Q}$-vector spaces $H^k_{dR}(X,Y)$ and $H_k^B(X,Y)$ are finite-dimensional, and the integration pairing is non-degenerate.

Example

$X = \mathbb{G}_m$ and $Y := \{1, a\}$ with $\mathbb{Q} \ni a \neq 1$. With work, one finds bases:

$$H^1_{dR}(X,Y) \supset \{[dx], [dx/x]\} \qquad H_1^B(X,Y) \supset \{[1 \to a], [\text{unit circle}]\}$$

$$A := \begin{pmatrix} \int_1^a dx & \int_1^a \frac{dx}{x} \\ \oint dx & \oint \frac{dx}{x} \end{pmatrix} = \begin{pmatrix} a-1 & \log(a) \\ 0 & 2\pi i \end{pmatrix} \qquad \det(A) = 2\pi i(a-1) \neq 0$$

Modern approach: motives [Grothendieck, Voevodsky, ...]

- $(X, Y) \ncong (X', Y')$ may have the same (co)homology/periods
- Associate $(X, Y) \mapsto M(X, Y)$, the "motive of (X, Y)"
  - fundamental mathematical object analogous to a group/vector space/...
  - encodes all (co)homological information about (X, Y)
  - respects obvious rules like Fubini, Stokes, etc.
  - forgets all other information
- Strategy: prove theorems about periods by manipulating motives

Example: $M(X, Y)$ is "mixed Tate over $\mathbb{Z}$" if, roughly speaking, points do not disappear when we reduce modulo a prime

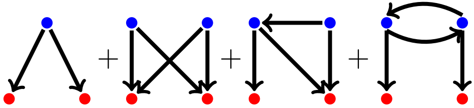
e.g. $(\mathbb{G}_m, \{1\})$ but not $(\mathbb{G}_m, \{1, 2\})$ since $2 \equiv 0 \mod 2$

Theorem (Brown 2012): this implies that all periods of (X, Y) are $\mathbb{Q}[2\pi i]$-linear combinations of "multiple zeta values"

$$\zeta(k_1, \ldots, k_d) := \sum_{n_1 > \cdots > n_d > 0} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

Current research

- Going beyond "mixed Tate over $\mathbb{Z}$"
- Applications to particle physics (Broadhust–Kreimer, ...)

  observable quantity $\sim$  $+ \cdots$

  where each diagram corresponds to a period integral
- Special values of L-functions, generalizing $\zeta(n)$
- Motivic Galois group: hidden symmetry of $\mathscr{P}$, generalizing $i \mapsto -i$
- ...

Thank you!