What's the deal with homological algebra?

Part 1: chain complexes + topological spaces

Betti (1870s) + Noether (1920s) + lots of mathematicians (1950s)

topological spaces $\longrightarrow$ algebra

$\rightsquigarrow$ groups

Good way to understand / build spaces:

n-dimensional pieces

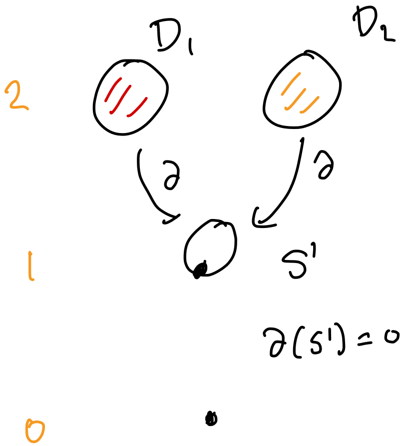
$\partial \downarrow$

n-1-dimensional pieces

abelian group

$\downarrow \partial$

ab. group



$\partial(S^1) = 0$

$$\begin{array}{c} \mathbb{Z}^2 \\ \big\downarrow (1\ 1) \\ \mathbb{Z} \\ \big\downarrow 0 \\ \mathbb{Z} \end{array}$$

<u>Key property</u>: $\partial(\partial(x)) = 0$ "$\partial^2 = 0$"

$B^3 = \{ v \in \mathbb{R}^3 \mid \|v\| \leq 1 \}$ ball 

$\downarrow \partial$

$S^2 = \{ v \in \mathbb{R}^3 \mid \|v\| = 1 \}$ sphere 

Aside:

$\partial(X \times Y) = \partial(X) \times Y \cup X \times \partial(Y)$ ⟷ in chain complexes: $\partial(cd) = \partial(c)\cdot d + c \cdot \partial(d)$ ⇒ differential graded algebra.

$X =$  $Y =$  $X \times Y =$ 

<u>Defn</u> a chain complex C is a sequence:

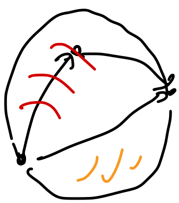
$$\cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

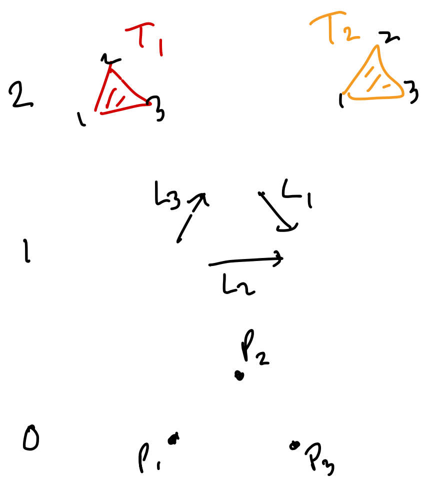
- each C_n is an ~~abelian group~~ vector spaces

- $\partial_i : C_i \to C_{i-1}$ maps of ~~abelian groups~~ vector spaces.

- such that $\partial_i \partial_{i+1}(c) = 0 \quad \forall c \in C_{i+1}$

Ex sphere again (using triangles)





$\longrightarrow$

$$
\begin{array}{ccc}
C_2 & & \mathbb{Z}^2 \\
\Big\downarrow \partial & & \Big\downarrow \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix} \\
C_1 & & \mathbb{Z}^3 \\
\Big\downarrow \partial & & \Big\downarrow \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
C_0 & & \mathbb{Z}^3
\end{array}
$$

- $\partial(T_1) = L_1 - L_2 + L_3 = \partial(T_2)$
  - $\partial(L_1 - L_2 + L_3) = ??$
- $\partial(L_1) = P_3 - P_2$

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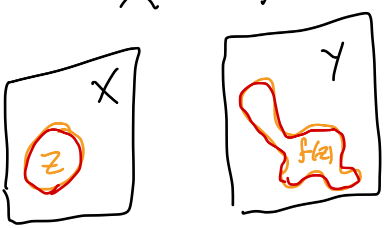
to fix the problem of non-uniqueness

$\longrightarrow$ need to talk about homotopy:

$X \xrightarrow{f} Y$ $\qquad \longleftrightarrow \qquad$ $C \xrightarrow{f} D$



a <u>chain map</u> is a map $f_i : C_i \longrightarrow D_i \quad \forall i$

such that

$$\partial(f_i(c)) = f_{i-1}(\partial c)$$

"$\partial f(z) = f(\partial z)$"

$X \overset{f}{\underset{g}{\rightrightarrows}} Y$ two cont maps

are homotopic if there is

$h: I \times X \to Y$ cont.

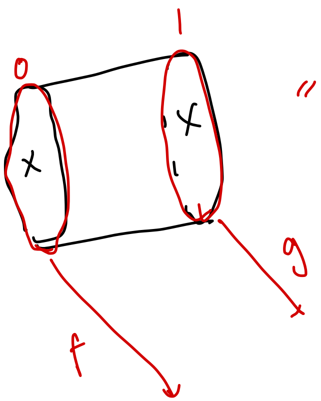
$I = [0,1]$

such that

$h(0,x) = f(x)$

$h(t,x) = ?$

$h(1,x) = g(x)$



" $\partial(h) = g - f$ "

a homotopy h

$h: C \to D$

$h_n: C_n \to D_{n+1} \quad \forall n$

$\forall c \in C$

$\partial h(c) + h\partial(c) = g(c) - f(c)$

(homotopy from f to g)

two chain maps

Eg fundamental group of $x \in X$ fixed

$$\pi_1(X) = \mathrm{map}_*(S^1, X)/\text{homotopy}.$$

$$\begin{tikzcd}
 & S^1 \arrow[d] & C \arrow[d] & \\
 & & \vdots \arrow[d, "\partial"] & \\
3 & 0 \arrow[r] \arrow[d] & C_3 \arrow[d, "\partial"] & \\
2 & 0 \arrow[r] \arrow[d] & C_2 \arrow[d, "\partial"] & \\
1 & \mathbb{Z} \arrow[r, "f"] \arrow[ur, "h"] \arrow[d] \arrow[dr, bend right, "0"] & C_1 \arrow[d, "\partial_1"] & \\
0 & 0 \arrow[r] & C_0 &
\end{tikzcd}$$

$\in C_2$

$\boxed{h(1) = d}$?

$\Longleftrightarrow$ $f(1) = c \in C_1$

chain map

$\partial(c) = 0$

$\partial(f(x)) = f(\partial(x))$

$\mathrm{map}(S^1, X)/\sim \;\longleftrightarrow\;$ elts $c \in C_1$ s.t. $\partial(c) = 0 /\sim$?

equivalence relation

$c = c'$

if $c - c' = \partial_2(d)$

$d \in C_2$

in the end

i.e. $\{c \in C_1 \mid \partial_1(c) = 0\}$

$$\pi_1(X) \longleftrightarrow \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = H_1(C)$$

quotient group.

"homology group"

in fact: $H_n(C) = \dfrac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$

nth homology group.

Ex's both spheres:

$$
\begin{array}{ll}
2 & \mathbb{Z}^2 \\
 & \downarrow \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix} \\
1 & \mathbb{Z}^3 \\
 & \downarrow \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
0 & \mathbb{Z}^3
\end{array}
$$

$\begin{pmatrix} a \\ -a \end{pmatrix}$

$H_2 \cong \ker \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix} \cong \mathbb{Z}$

$H_1 \cong 0$

$H_0 = \mathbb{Z}^3 / \operatorname{im} \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cong \mathbb{Z}$

$$
\begin{array}{ll}
 & 0 \\
 & \downarrow \\
2 & \mathbb{Z}^2 \\
 & \downarrow (1\ 1) \\
1 & \mathbb{Z} \\
 & \downarrow 0 \\
0 & \mathbb{Z}
\end{array}
$$

$\begin{pmatrix} a \\ -a \end{pmatrix}$

$H_2(C) = \ker(1\ 1) / 0 \cong \mathbb{Z}$

$H_1(C) = \mathbb{Z} / \operatorname{im}(1\ 1) \cong 0$

$H_0(C) = \mathbb{Z} / \operatorname{im}(0) \cong \mathbb{Z}$