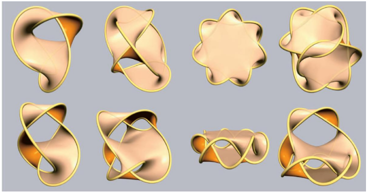
Seifert Surfaces and Knot Genus

Alex Teeter

University of Toronto Scarborough

March 30th 2022

The Plan



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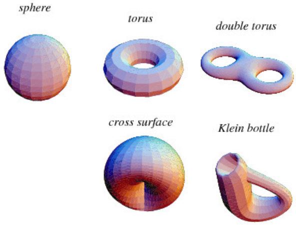

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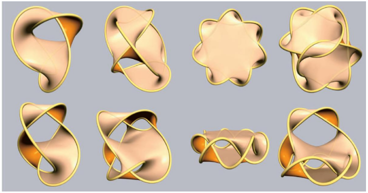

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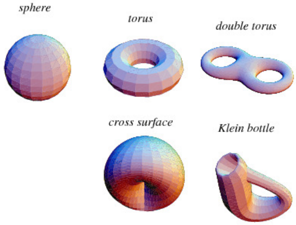

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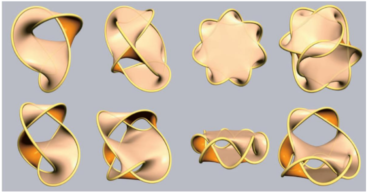

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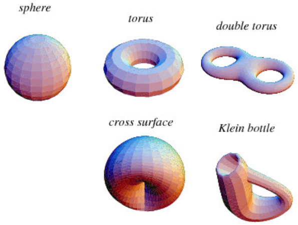

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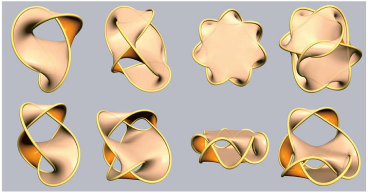

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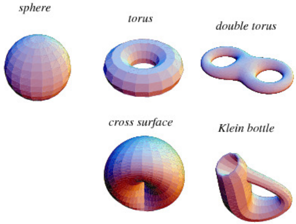

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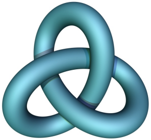

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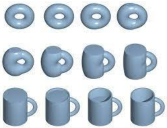

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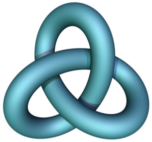

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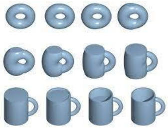

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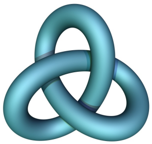

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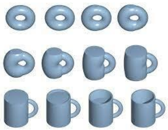

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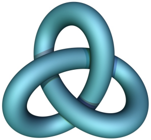

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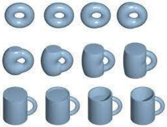

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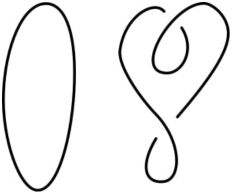

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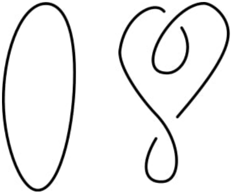

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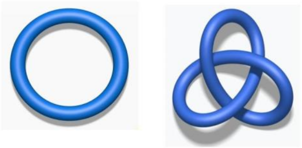

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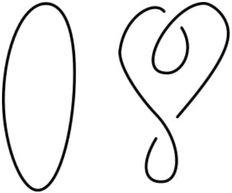

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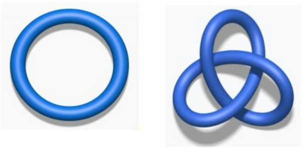

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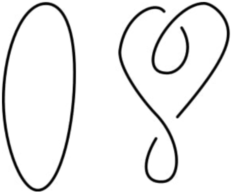

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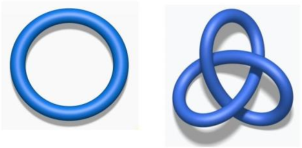

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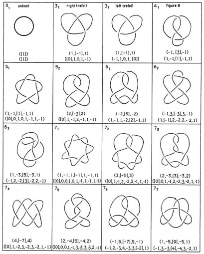

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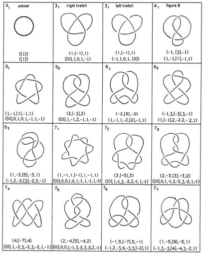

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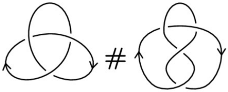

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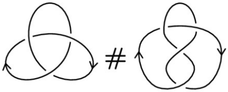

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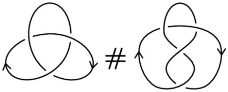

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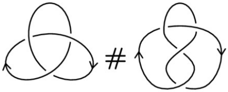

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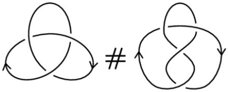

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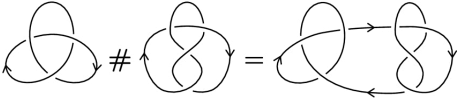

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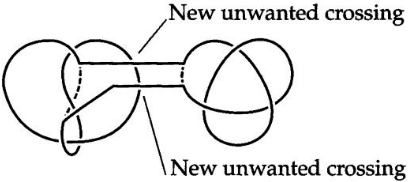

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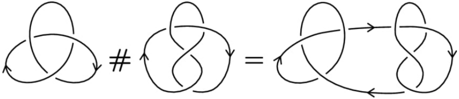

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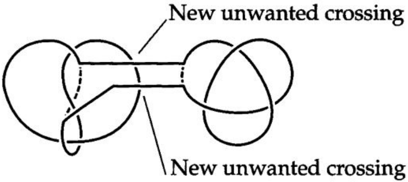

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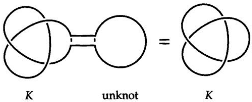
Prime Knots



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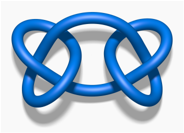

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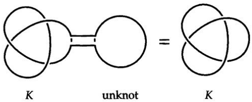

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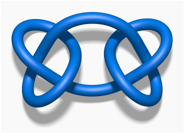

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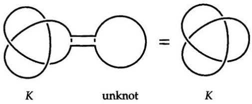

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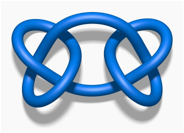

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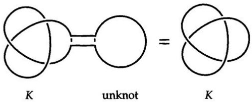

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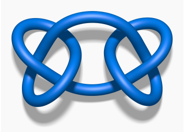

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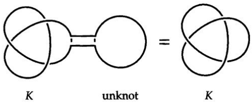

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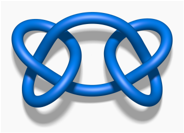

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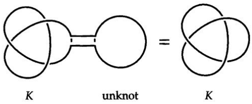

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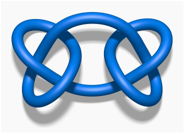

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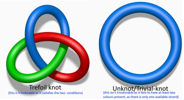

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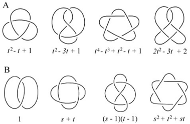

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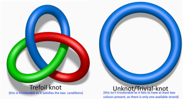

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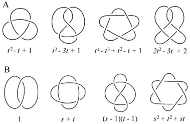

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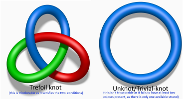

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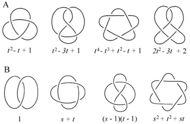

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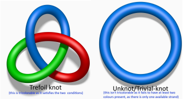

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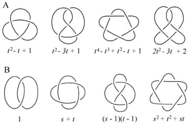

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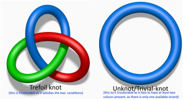

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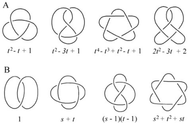

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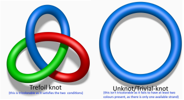

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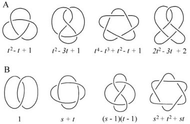

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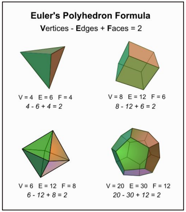

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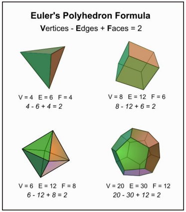

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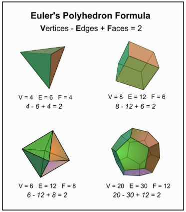

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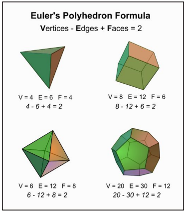

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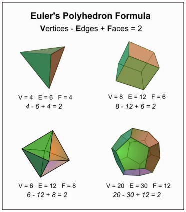

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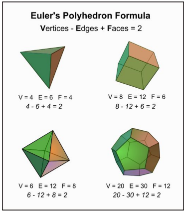

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- We will see this formula also generalize when we cover topological surfaces

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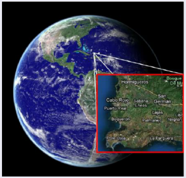

Figure: The surface of the earth is a topological surface: flat locally, spherical globally!

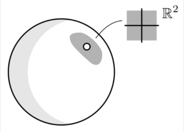

Figure: Every point in the surface must have a neighborhood that resembles the plane (look flat locally).

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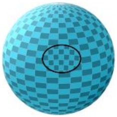

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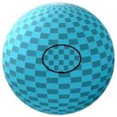

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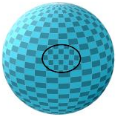

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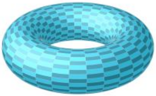

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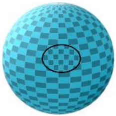

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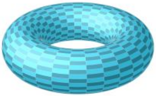

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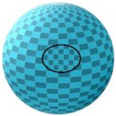

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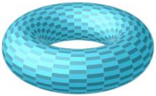

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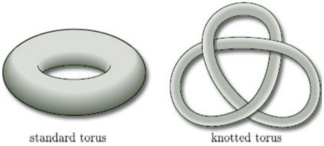

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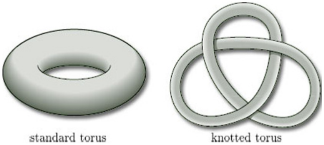

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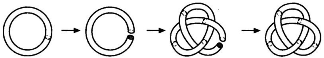

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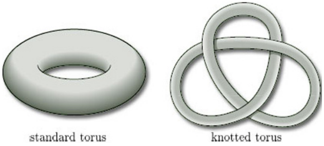

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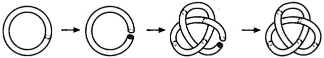

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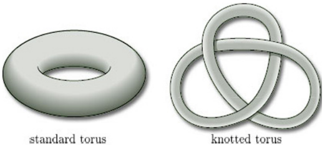

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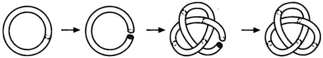

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- For those who have taken a linear algebra or abstract algebra course, you can think of them as the isomorphisms of topology

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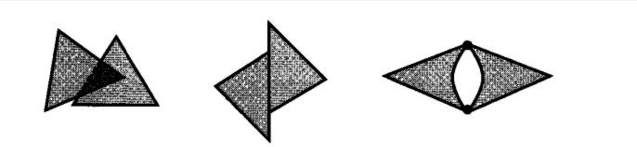

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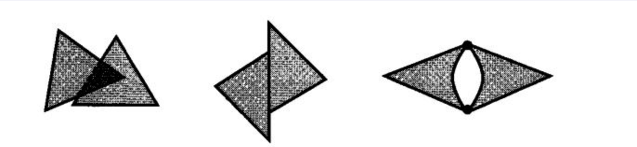

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for any (finite) triangulation τ of Σ

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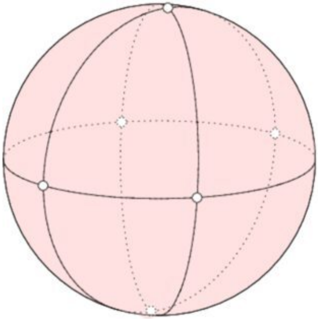

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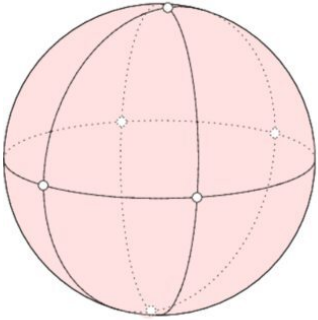

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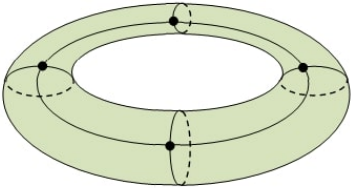

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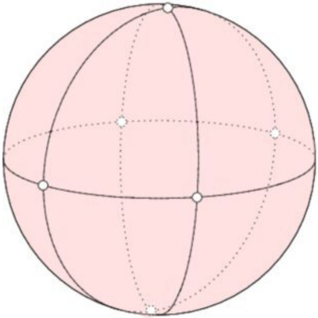

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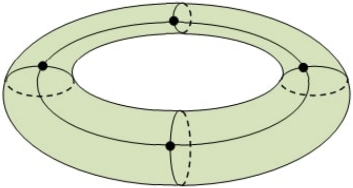

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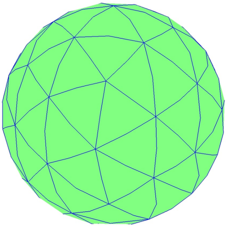

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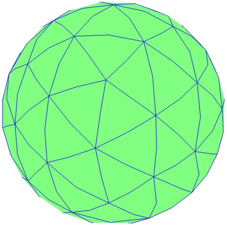

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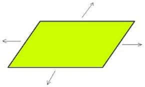

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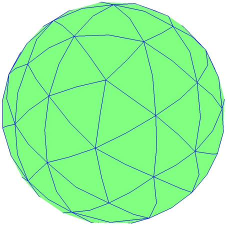

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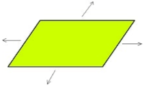

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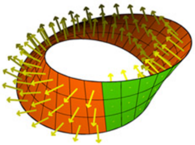

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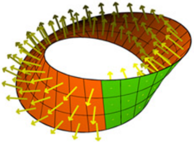

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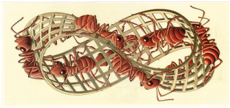

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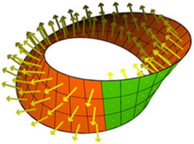

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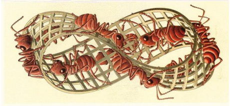

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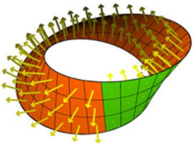

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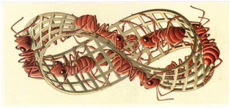

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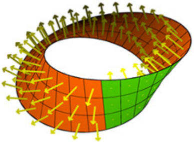

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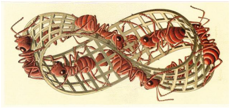

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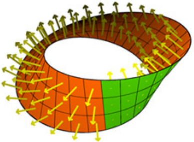

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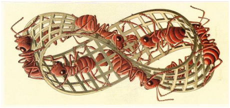

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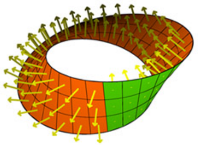

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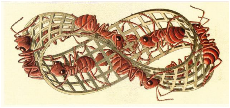

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- These surfaces are interesting in their own right, however, for our purposes, we will prefer our surfaces to be orientable.

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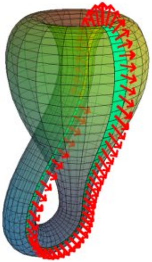

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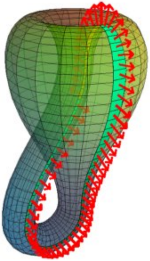

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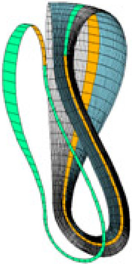

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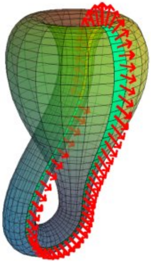

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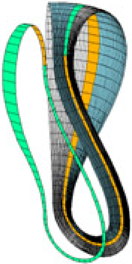

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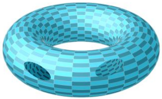

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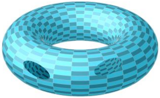

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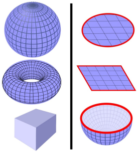

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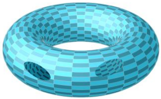

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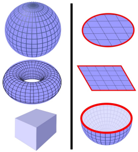

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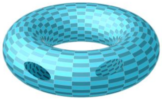

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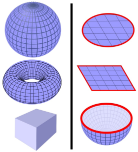

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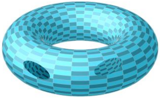

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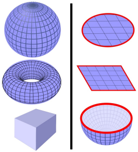

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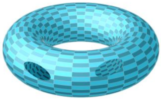

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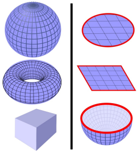

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- We will be investigating such surfaces when we get back to knots.

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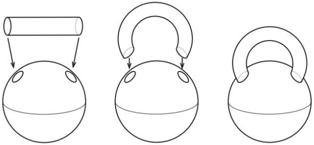

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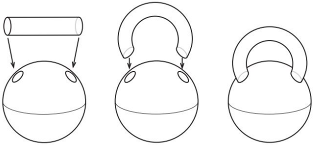

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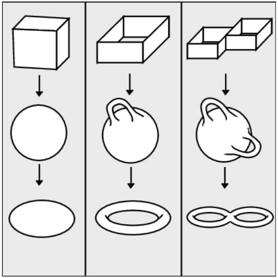

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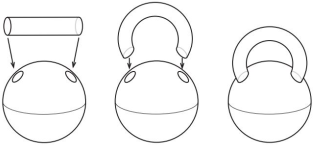

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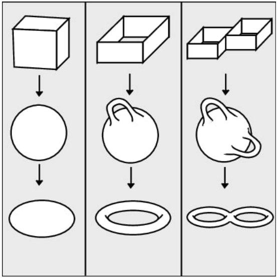

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as wanted! This completes the proof! ∎

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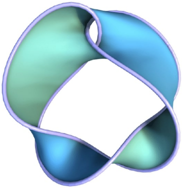

Figure: A compact orientable surface bounded by the figure 8 knot. If we triangulate it and compute its Euler Characteristic, we would see that it is -1. Using our identity $\chi(\Sigma) = 2 - 2g$, and capping off the 1 boundary component with a disk (adding 1 to -1, we get that this surface has genus 1: i.e it is homeomorphic to a torus with 1 boundary component removed.

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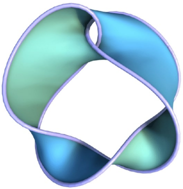

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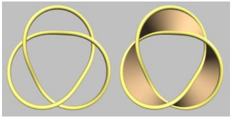

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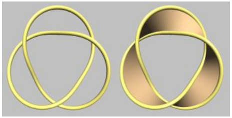

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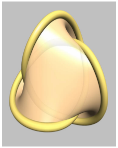

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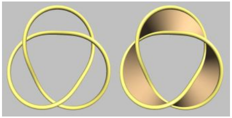

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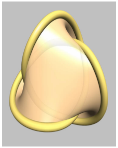

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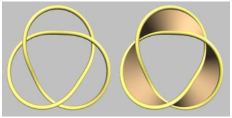

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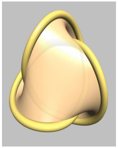

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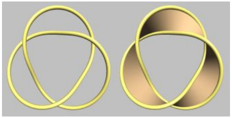

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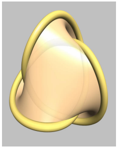

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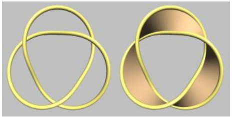

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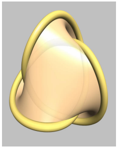

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- Issue: Sometimes we can end up with a non-orientable surface using this method.

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Seifert's algorithm comes to the rescue!

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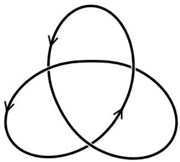

Figure: We begin the algorithm by picking an orientation to travel around the Knot.

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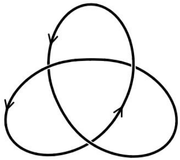

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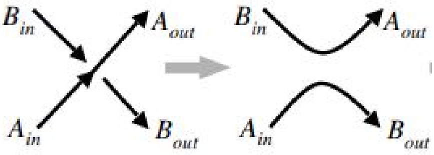

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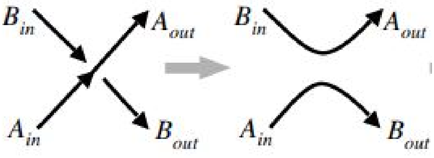

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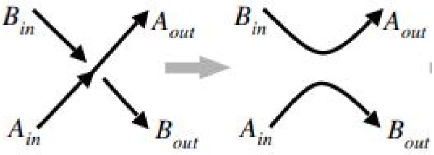

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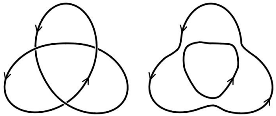

Figure: Performing this algorithm on the trefoil, we would get disks: these are our seifert circles

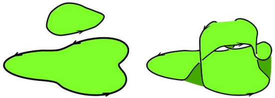

Figure: We finish off by attaching each crossing with bands with a half-twist

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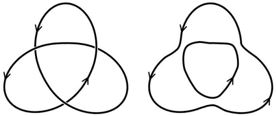

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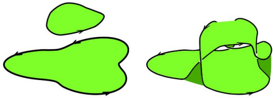

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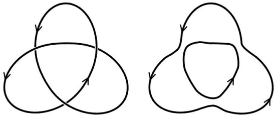

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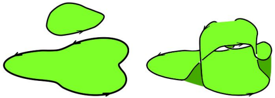

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- 7. We have obtained our Seifert Surface for K! (We have a compact orientable surface bounded by K!)

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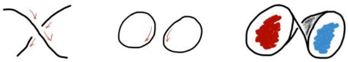

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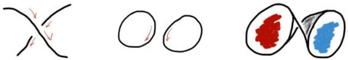

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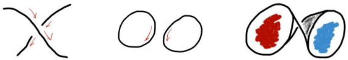

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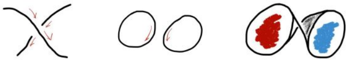

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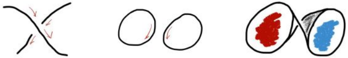

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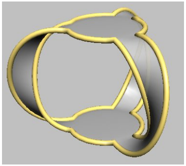

Figure: The Seifert Surface we constructed for the trefoil, before isotoping it

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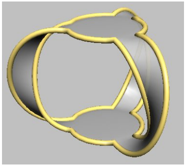

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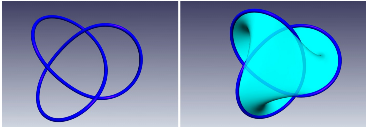

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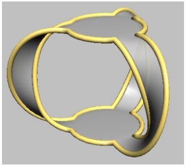

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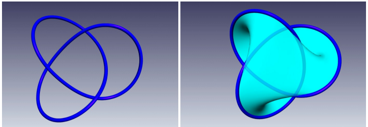

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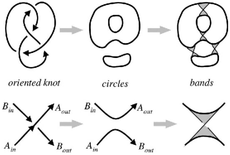

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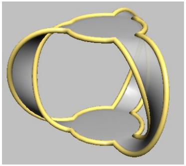

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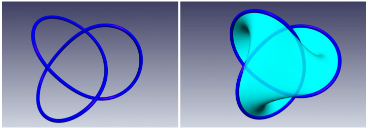

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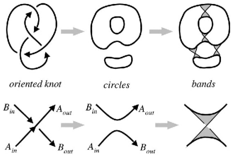

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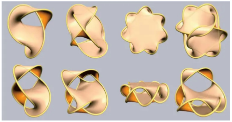

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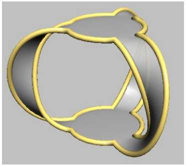

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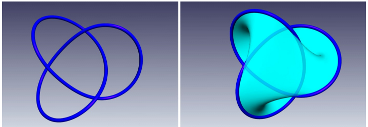

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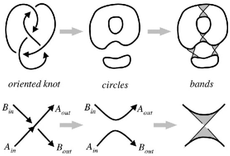

Figure: Another construction of a Seifert Surface: This time for a figure eight knot

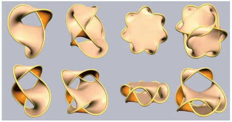

Figure: Seifert surfaces of various Knots

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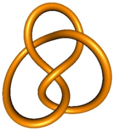

Figure: Alternating projection of the figure eight knot

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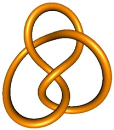

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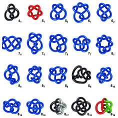

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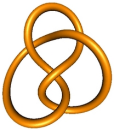

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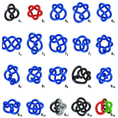

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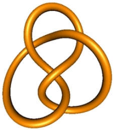

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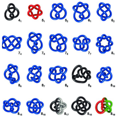

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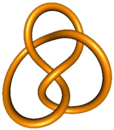

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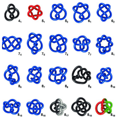

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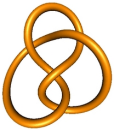

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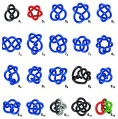

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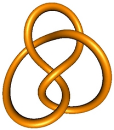

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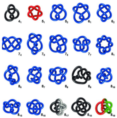

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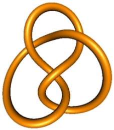

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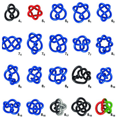

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A Knot K is the Unknot if and only if its minimal genus is 0 (i.e it bounds a disk)

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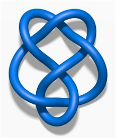

Figure: The 6_2 Knot

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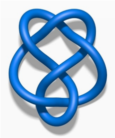

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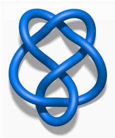

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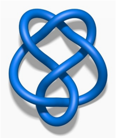

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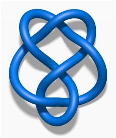

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Hope this gives you a good idea of how we can use our Invariant to distinguish between Knots!

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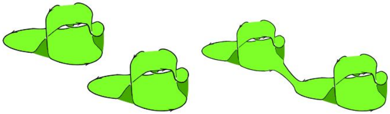

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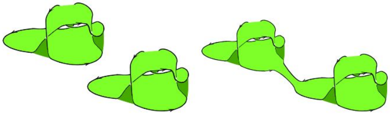

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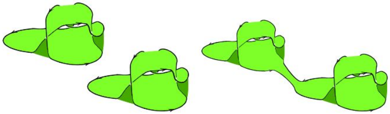

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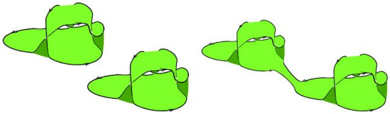

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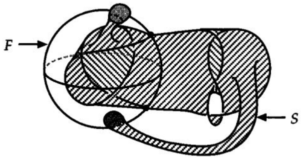

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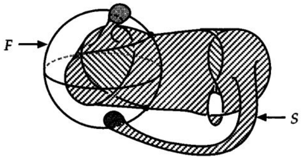

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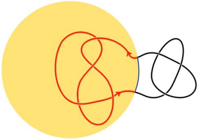

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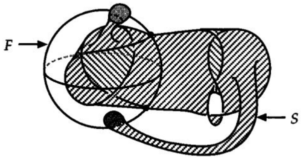

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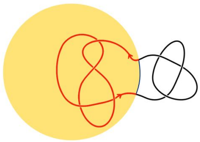

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- We consider the intersection arc β that runs from the intersection points of S and F (Where the punctures are) and note that $\beta \cup \alpha_1 = K_1$ and $\beta \cup \alpha_2 = K_2$.

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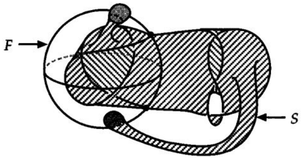
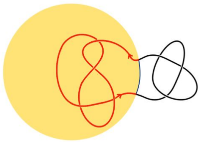


Figure: We seperate the K_1 and K_2 part of our surface with F



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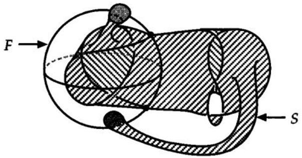

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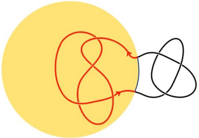

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- We want to construct a surface such that F cleanly divides it into K_1 and K_2 along β (i.e there are no other intersections)

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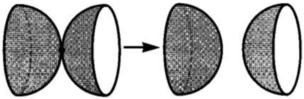

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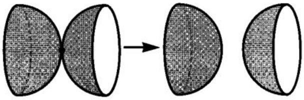

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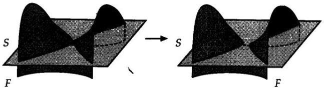

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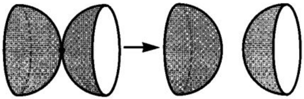

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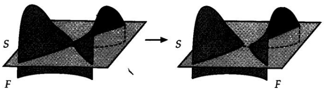

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- So the only elements of $S \cap F$ should consist of the curve β and Loops. (Note: β is the only curve since ∂S only intersects F at the 2 points in the Knot Sum)

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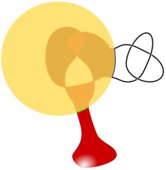

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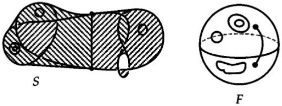

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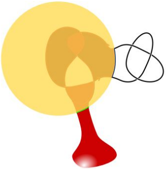

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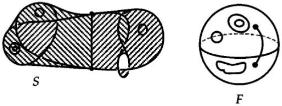

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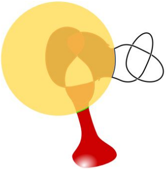

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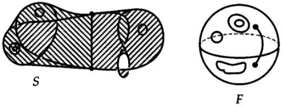

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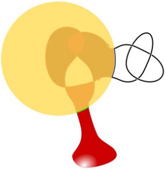

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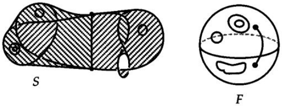

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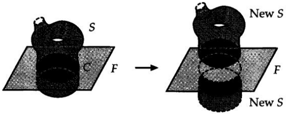

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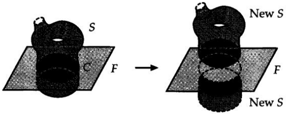

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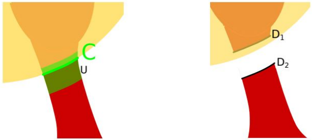

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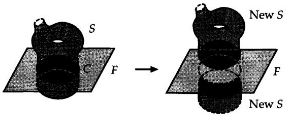

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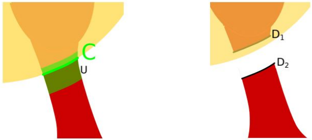

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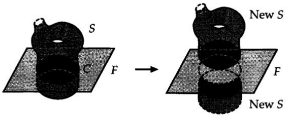

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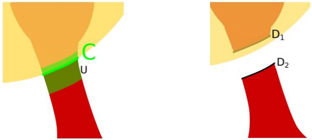

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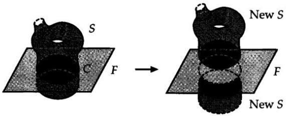

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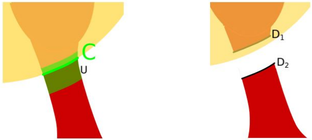

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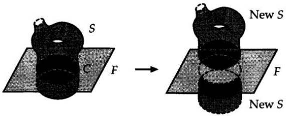

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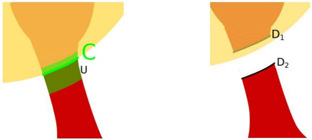

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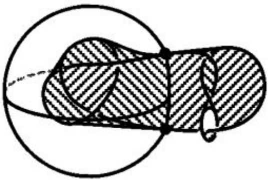

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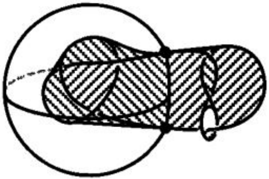

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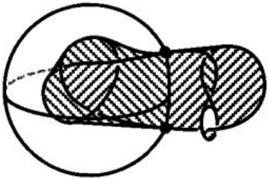

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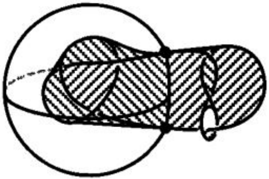

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- These are not necessarily minimal, so $g(K_1) + g(K_2) \leq g(K_1 \# K_2)$ as wanted! ■

Other Results

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Theorem

If $g(K) = 1$, then K is a prime knot

- Assume $K = K_1 \# K_2$. (Where K_1 and K_2 are nontrivial) But then

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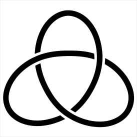

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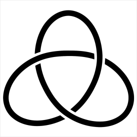

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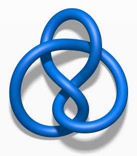

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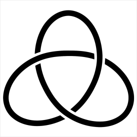

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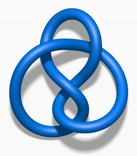

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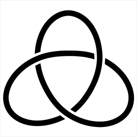

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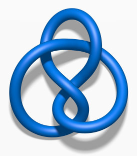

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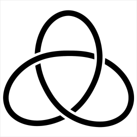

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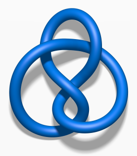

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Theorem

Every Knot is a finite sum of prime Knots

Follows from $g(K_1 \# K_2) = g(K_1) + g(K_2)$ and genus 1 knots guaranteed prime.

Conclusion

Figure: Our Friends: The compact orientable surfaces :)

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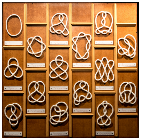

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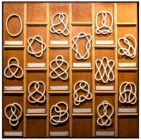

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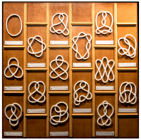

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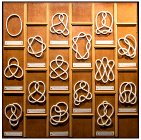

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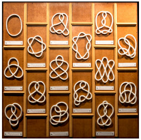

Figure: Hopefully this encouraged you to do further reading on Knot theory and Topology! This is "Knot" the end ;)

- Not only have we found a useful invariant for differentiating between Knots, but we have also managed to use it to prove important theorems about Knot Theory and Prime Knots!
- Hope this helped demonstrate the power of surface topology/classification of surfaces theorem and its beautiful connection to Knot Theory!
- Hope you enjoyed the presentation!

Table of Contents

Further Reading

- The Knot Book - Collin Adams (Very nice introduction to Knot Theory, no prerequisites needed but covers some very deep ideas still, fun and inspiring read)
- Knot Theory - Math at Andrews Lecture Series: (A fantastic lecture series, doesn't assume much background other than some basic Linear Algebra and a tiny bit of group theory, comes with exercises)
- Euler's Gem - David S. Richeson (Fantastic book that goes over the history and concepts of Topology: assumes no prior background and is a very fun read)
- An Introduction to Knot Theory - W.B.R Lickorish (Another great text on Knot Theory, a lot more heavy than Colins Book, some algebraic Topology background is assumed)
- Topology - Munkres (The golden standard for an introductory course to topology, no background is technically needed but is good to know some analysis, covers classification of surfaces in much more detail)

References

- 1. Adams, Collin. The Knot Book. 2nd ed., AMS, 2004.
- 2. Wang, Mengtong. Introduction to Seifert Surfaces and Their Properties. University of Sydney, https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.639.3970am =rep1amp;type=pdf.
- 3. Rodriguez, Eric Vallespi. Knots and Seifert Surfaces, University of Barcelona, http://diposit.ub.edu/dspace/bitstream/2445/178980/2/178980.pdf.
- 4. Birdman, John S. "New Points of View in Knot Theory ." MAA.org, Bulletin of the AMS, 1993, https://www.maa.org/programs/maa-awards/writing-awards/new-points-of-view-in-knot-theory.
- 5. van Wjik, Jarke J., and Arjeh M. Cohen. Visualization of Seifert Surfaces. Technische Universiteit Eindhoven, Aug. 2006, https://www.maths.ed.ac.uk/ v1ranick/papers/vanwijk.pdf.

References

- 6. Richeson, David. Euler's Gem: The Polyhedron Formula and The Birth of Topology. 2nd ed., Princeton University Press.
- 7. Knot Theory- Math at Andrews. Andrews University, 1 May 2019, https://www.youtube.com/playlist?list=PLOROtRhtegr4c1H1JaWN1f6J_q1HdWZOY.
- 8. Livingston, Charles. Knot Theory. Indiana University.
- 9. Bar-Natan, Dror. "MAT1350 - Knot Theory Notes (Uoft)." Math 1350F - Knot Theory, University of Toronto, https://www.math.toronto.edu/ drorbn/classes/0304/KnotTheory/index.html.
- 10. Lickorish, W.B.R. An Introduction to Knot Theory. Springer, 1997.
- 11. Gallier, Jean, and Dianna Xu. A Guide to the Classification of Compact Surfaces. Springer, 2013.