

Seifert Surfaces and Knot Genus

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University of Toronto Scarborough

March 30th 2022

The Plan

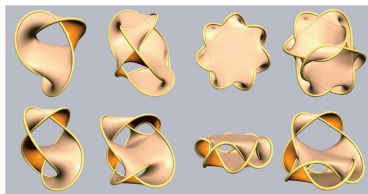


Figure: Seifert Surfaces of various Knots

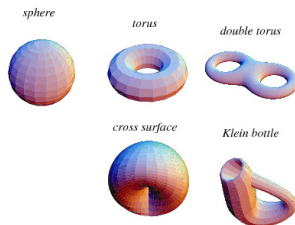


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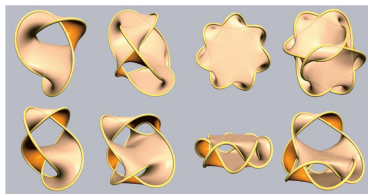


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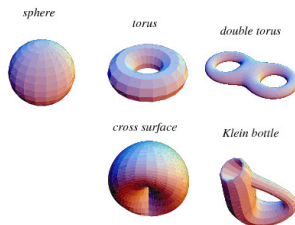


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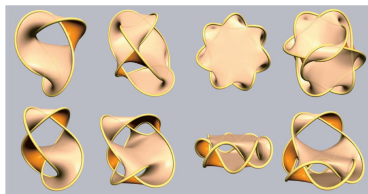


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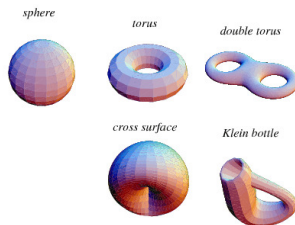


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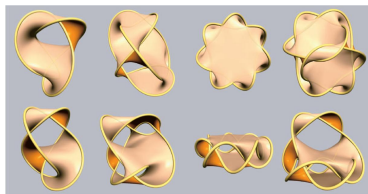


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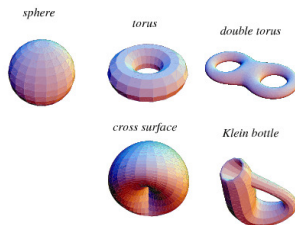


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- 2 Topological Surfaces
- 3 Seifert Surfaces
- 4 Important Proof and other results
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What is Knot Theory?

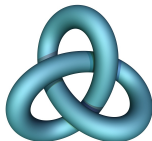


Figure: The Trefoil Knot in \mathbb{R}^3



Figure: To a Topologist, the donut and coffee mug are equivalent under continuous mapping/deformation

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A Knot K is a projection/image of S_1 (a unit circle) embedded (placed in) \mathbb{R}^3 (3-D Space).

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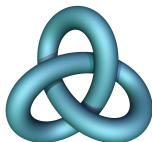


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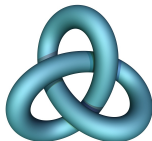


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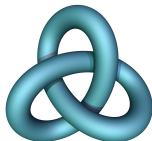


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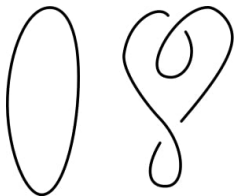


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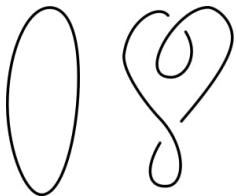


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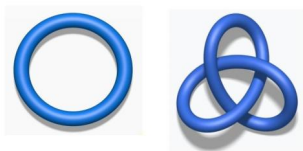


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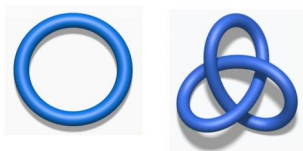


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We say 2 knots K_1 and K_2 are equivalent ($K_1 = K_2$) if there exists an ambient isotopy (continuous deformation over time) $H : \mathbb{R}^3 \times [0;1] \rightarrow \mathbb{R}^3$ and a continuous bijection (homeomorphism) $H_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ s.t. $H_1 = H(K_1;1) = K_2$. (We can think of $[0;1]$ representing time, and us continuously deforming K_1 into K_2 through 3-D space over time).

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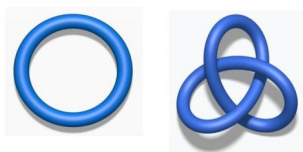


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













S_1 unknot  R1B R1B	S_2 trefoil knot  R1B-R1B R1B-R1B-R1B	S_3 full twist  R1B-R1B R1B-R1B-R1B	S_4 figure 8  R1B-R1B-R1B R1B-R1B-R1B
S_5  R1B-R1B-R1B R1B-R1B-R1B-R1B	S_6  R1B-R1B R1B-R1B-R1B-R1B	S_7  R1B-R1B-R1B R1B-R1B-R1B-R1B	S_8  R1B-R1B-R1B-R1B R1B-R1B-R1B-R1B
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S_{13}  R1B-R1B R1B-R1B-R1B-R1B	S_{14}  R1B-R1B-R1B R1B-R1B-R1B-R1B	S_{15}  R1B-R1B-R1B R1B-R1B-R1B-R1B	S_{16}  R1B-R1B-R1B R1B-R1B-R1B-R1B

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













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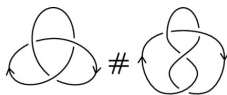


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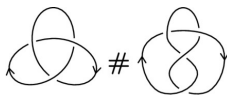


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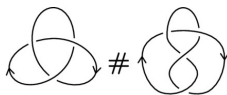


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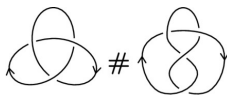


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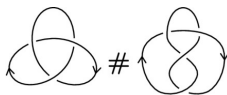


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The sum of Knots K_1 and K_2 , $K_1 \# K_2$, is the gluing of $K_1; K_2$ along a section of both such that no new crossings are added.

Figure: A valid Knot Sum

Figure: Not a valid Knot Sum (Pun intended)

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If the orientation of the 2 knots are the same, then we get 1 possible knot from the composition. If the orientation is different between the 2, we could potentially get a different knot (though there is a case where the sums are the same).

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- We will see this formula also generalize when we cover topological surfaces

Definition

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For every point $p \in X$, there is an open neighborhood U of p and a homeomorphism (bicontinuous bijection) from U to an open set of \mathbb{R}^2 . (X must also be 2nd countable and Hausdorff, however we will not go into such technical terms)

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Figure: The surface of the earth is a topological surface: locally, spherical globally!

Figure: Every point in the surface must have a neighborhood that resembles the plane (look at locally).

Some Examples

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- Homeomorphisms preserve many of the properties we like in Topology
- For those who have taken a linear algebra or abstract algebra course, you can think of them as the isomorphisms of topology

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1. $\bigcup_{i \in I} T_i = S$
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Thus, it naturally makes sense to give the following definition of Euler Characteristic of a surface below

The Euler characteristic of a surface (S) is

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for any (finite) triangulation of S .

Figure: This triangulation of the sphere shows us that the Euler characteristic of the sphere is $V - E + F = 2$ after counting up all the vertices, edges and faces. Note this is the same for anything homeomorphic to the sphere and for any triangulation of the sphere

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These surfaces are interesting in their own right, however, for our purposes, we will prefer our surfaces to be orientable.

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- Something to note: adding in d boundary components decreasing the Euler Characteristic by d (it removes a face homeomorphic to a triangle, but doing so does not remove any edges or vertices)
- We will be investigating such surfaces when we get back to knots.

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Corollary

For a given compact orientable surface (without boundary) Σ ,
 $\chi(\Sigma) = 2 - 2g$, where g is the genus of Σ

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We prove this by induction on g .

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Inductive Step: Suppose for a surface with g handles, we have $\chi(S) = 2 - 2g$

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as wanted! This completes the proof!

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Figure: A compact orientable surface bounded by the figure 8 knot. If we triangulate it and compute its Euler Characteristic, we would see that it is $\chi = 2 - 2g$. Using our identity $\chi = 2 - 2g$, and capping off the 1 boundary component with a disk (adding 1 to χ), we get that this surface has genus 1: i.e it is homeomorphic to a torus with 1 boundary component removed.

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- 4 Important Proof and other results
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- Issue: Sometimes we can end up with a non-orientable surface using this method.

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Figure: Performing this algorithm on the trefoil, we would get disks: these are our Seifert circles

5. Performing this algorithm, we produce disks (some disks are on top of each other, this is fine). We call these disks the Seifert circles.

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7. We have obtained our Seifert Surface Σ ! (We have a compact orientable surface bounded by K !)

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We also have the following theorem:

A Knot K is the Unknot if and only if its minimal genus is 0 (i.e it bounds a disk)

Figure: The 6_2 Knot

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Hope this gives you a good idea of how we can use our Invariant to distinguish between Knots!

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- 2 Topological Surfaces
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We can simply join the 2 surfaces through a single band, creating a Seifert Surface for $K_1 \# K_2$ whose genus is the sum $g(K_1) + g(K_2)$. So either the minimal genus of $K_1 \# K_2$ is $g(K_1) + g(K_2)$, or there is a surface of smaller genus. We will show there isn't.

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We consider a minimal genus Seifert Surface for $K_1 \# K_2$. Since $K_1 \# K_2$ is a composite Knot, we can consider a twice punctured Sphere F that separates $K_1 \# K_2$, the boundary of S , into 2 arcs γ_1 and γ_2 at the punctured points given by the Knot Sum definition.

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We consider the intersection arc that runs from the intersection points of S and F (Where the punctures are) and note that

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So in $\mathbb{R}^3 \setminus F$, there is the K_2 portion of the surface, and inside F there is the K_1 portion.

We want to construct a surface such that F cleanly divides it into K_1 and K_2 along γ (i.e there are no other intersections)

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So the only elements of $S \setminus F$ should consist of the curve and Loops. (Note: ∂S is the only curve since S only intersects F at the 2 points in the Knot Sum)

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Figure: Intersection loops and on S and F

To remove the intersection loops, we will "perform surgery" on S (Surgery techniques like these are common in Topology and Manifolds)!

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Since we added 2 disks D_1 and D_2 , and $\chi(D) = 1$ (for a disk), if $S \cup C$ is connected, then we have a contradiction as we increased the Euler Characteristic by 2 and thus lowered the genus, so wouldn't be minimal genus.

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Repeating this process nitely many times until all Loops are removed, we obtain a Minimal Seifert Surface S for $K_1 \# K_2$ that is divided into a K_1 Seifert Surface and K_2 Seifert Surface by S and has no other intersections.

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Figure: Yay! S divides cleanly through F into K_1 and K_2 !!

- Throwing away this part will leave the genus the same, as we are keeping D_1 but throwing away D_2 .
- Repeating this process finitely many times until all Loops are removed, we obtain a Minimal Seifert Surface S^∂ for $K_1 \# K_2$ that is divided into a K_1 Seifert Surface and K_2 Seifert Surface by S and has no other intersections.
- These are not necessarily minimal, so $g(K_1) + g(K_2) \geq g(K_1 \# K_2)$ as wanted!

Theorem

The Unknot is not the Sum of 2 non-trivial Knots

- Assume that the Unknot, O , is equivalent to $K_1 \# K_2$, where K_1 and K_2 are not the Unknot

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Theorem

If $g(K) = 1$, then K is a prime knot

- Assume $K = K_1 \# K_2$. (Where K_1 and K_2 are nontrivial) But then

$$g(K) = g(K_1 \# K_2) = g(K_1) + g(K_2) = 1$$

Other Results

Figure: Trefoil Knot has Genus one as we have shown, so it must be prime

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Note: since we found that the Trefoil and Figure Eight Knot have Genus 1, they are both prime :D

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Figure: Trefoil Knot has Genus one as we have shown, so it must be prime

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which implies $K_1 = \emptyset$ or $K_2 = \emptyset$, a contradiction! So K is prime!

Note: since we found that the Trefoil and Figure Eight Knot have Genus 1, they are both prime :D

Every Knot is a finite sum of prime Knots

Follows from $g(K_1 \# K_2) = g(K_1) + g(K_2)$ and genus 1 knots guaranteed prime.

Conclusion

Figure: Our Friends: The compact orientable surfaces :)

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Not only have we found a useful invariant for differentiating between Knots, but we have also managed to use it to prove important theorems about Knot Theory and Prime Knots!

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Table of Contents

- 1 An Intro to Knot Theory
- 2 Topological Surfaces
- 3 Seifert Surfaces
- 4 Important Proof and other results
- 5 References/Further Reading

Further Reading

- The Knot Book - Collin Adams (Very nice introduction to Knot Theory, no prerequisites needed but covers some very deep ideas still, fun and inspiring read)
- Knot Theory - Math at Andrews Lecture Series: (A fantastic lecture series, doesn't assume much background other than some basic Linear Algebra and a tiny bit of group theory, comes with exercises)
- Euler's Gem - David S. Richeson (Fantastic book that goes over the history and concepts of Topology: assumes no prior background and is a very fun read)
- An Introduction to Knot Theory - W.B.R Lickorish (Another great text on Knot Theory, a lot more heavy than Colins Book, some algebraic Topology background is assumed)
- Topology - Munkres (The golden standard for an introductory course to topology, no background is technically needed but is good to know some analysis, covers classification of surfaces in much more detail)

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