UTM C Undergrad Seminar

11/02/26

A Hitchhiker's Guide to Braids

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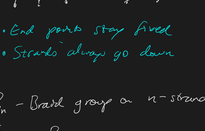
§1. Braids

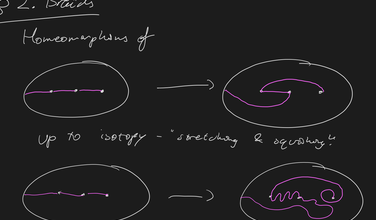


$\in B_3$ - 3-stranded braids.

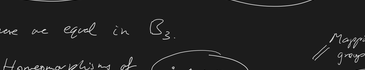
- End points stay fixed
- Strands always go down

B_n - Braid group on n-strands.

Eg. in B_4:

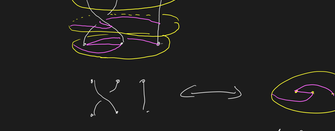


Identity:

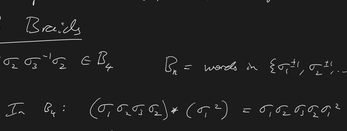


§2. Braids

Homeomorphisms of

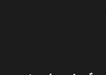


up to isotopy - "stretching & squishing"



These are equal in B_3.

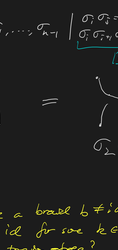
B_n = Homeomorphisms of

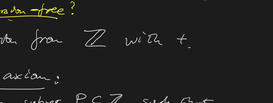


Mapping class group of

isotopy

Translation:





The operation is simply composition of functions.

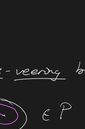
§3 Braids

$\sigma_1\sigma_2\sigma_3^{-1}\sigma_2 \in B_4$ $\quad B_n$ = words in $\{\sigma_1^{\pm1}, \sigma_2^{\pm1}, \dots, \sigma_{n-1}^{\pm1}\}$

Eg In B_4: $(\sigma_1\sigma_2\sigma_3\sigma_2) * (\sigma_1^{-2}) = \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1^{-2}$

$\sigma_1\sigma_2\sigma_3\sigma_2^{-1} = \sigma_1$.

$\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-2} \in B_3$. Translation

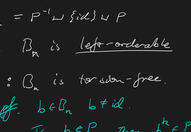


In B_4:

$\sigma_1\sigma_3 = \sigma_3\sigma_1$

$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } |i-j|\geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \forall i \in \{1, \dots, n-3\} \end{array} \right\rangle$

Braiding relation. Miraculously, these are all the rules you need!



$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$ABA = BAB$

Q. Is there a braid $b \neq id$ so that $b^k = id$ for some $k \in \mathbb{Z}$? ie is B_n torsion-free?

$\sigma_1\sigma_2^{-1}$

Take inspiration from $\mathbb{Z}$ with $+$.

An important axiom:

There is a subset $P \subset \mathbb{Z}$ such that

- If $a, b \in P$ then $a+b \in P$
- $\forall a \in \mathbb{Z}$, $a \in P$ or $-a \in P$ or $a = 0$ (exactly one is true)

$\mathbb{Z} = P \sqcup \{0\} \sqcup -P$.

We can now define $a < b$ iff $b - a \in P$.

$a+c < b+c$

Q. Can we do this for braids?

A. Yes!

Let $P \subset B_n$ be the set of right-veering braids

$\in P$

$\in P$

$\notin P$

Thm (Fen-Green-Rolfsen-Rourke-Wiest '92):

- If $x, y \in P$, then $xy \in P$
- $B_n = P^{-1} \sqcup \{id\} \sqcup P$

ie B_n is left-orderable

Cor: B_n is torsion-free.

pf. $b \in B_n$, $b \neq id$.

If $b \in P$, then $b^k \in P$ $\forall k > 0$.

Similar $b \in P^{-1}$ then $b^k \in P^{-1}$ $\forall k > 0$.

$x < y$ iff $x^{-1}y \in P$

$\Uparrow$

$zx < zy$ $\forall z \in B_n$