

Section 1: Complex Numbers

Remark: Where are we going? How will we get there?

The goal of this course is to get to Euler's theorem. We are going to do a very brief introduction to complex numbers and calculus. The class will be taught in an active learning style with lots of tasks. Please feel free to ask any questions. I hope that you enjoy the adventure.

Theorem: Euler's Theorem

If θ is a real number, then

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

**Activity:** Which value? Why?

09:15

(2 min.)

Find a value to plug in for θ to obtain:

$$e^{i\pi} + 1 = 0.$$

The Most
Beautiful
Equation.

Re-arrange Euler's equation to obtain this one.

$$\begin{aligned}\theta = \pi &\Rightarrow e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0 \\ &\Rightarrow e^{i\pi} + 1 = 0\end{aligned}$$

**Activity:** Name the parts.

09:18

(5 min.)

What are the constants e , i , π , 0 , and 1 ? How are they defined?

0 = zero = none

$$x + 0 = x$$

1 = one

$$1x = x$$

$\pi = 3.14 \dots$ "unending"

π = length of circle

$e = 2.71 \dots$ "natural constant"

length of diameter

$i = \sqrt{-1}$

"imaginary unit" $\ln(e^x) = x$

Definition: The Real and Complex Numbers

The **real numbers** are the familiar numbers expressible as decimals. We write $x \in \mathbb{R}$ for the statement " x is in the real numbers". The **complex numbers** are numbers of the form $a + bi$ where a and b are both real numbers. The **imaginary unit** i is defined so that: $i^2 = -1$. $\leftrightarrow i = \sqrt{-1}$


Activity: Calculate some values.

09:30

(5 min.)

Calculate the value of the following expressions.

1. $(1 - i)(1 + i) = 2$

2. $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3 = 1$

$$(1) \quad (1 - i)(1 + i) = 1^2 - i^2 = 1 - (-1) = 2$$

$$(2) \quad \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$$

$$= \left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2\left(-\frac{\sqrt{3}}{2}i\right) + 3\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}i\right)^2 + \left(-\frac{\sqrt{3}}{2}i\right)^3$$

$$= -\frac{1}{8} - \frac{3\sqrt{3}i}{8} + \cancel{\frac{9}{8}} + \frac{3\sqrt{3}i}{8}$$

$$= \frac{9-1}{8} = \frac{8}{8} = 1$$

$$\sqrt[3]{1} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Example: Powers of the Imaginary Unit

Consider the sequence of complex numbers $z_n = i^n$.

Write a simple pattern for the entries of this sequence.

Notice that the pattern repeats. How many terms does it take for the pattern to repeat?

$$\begin{array}{l} z_0 = i^0 = 1 \\ z_1 = i^1 = i \\ z_2 = i^2 = -1 \\ z_3 = i^3 = -i \\ \hline z_4 = i^4 = 1 \end{array}$$

$$z_5 = i^5 = i$$

$$z_6 = i^6 = -1$$

$$z_7 = i^7 = -i$$

\vdots

This pattern repeats.
It takes four terms
for the pattern to
repeat.

This gives

$$z_n = \begin{cases} 1 & n = 0, 4, 8, 12, \dots \\ i & n = 1, 5, 9, 13, \dots \\ -1 & n = 2, 6, 10, 14, \dots \\ -i & n = 3, 7, 11, 15, \dots \end{cases}$$

Definition: The Complex Plane

The complex plane $\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$ has a real axis \mathbb{R} and an imaginary axis $i\mathbb{R}$.

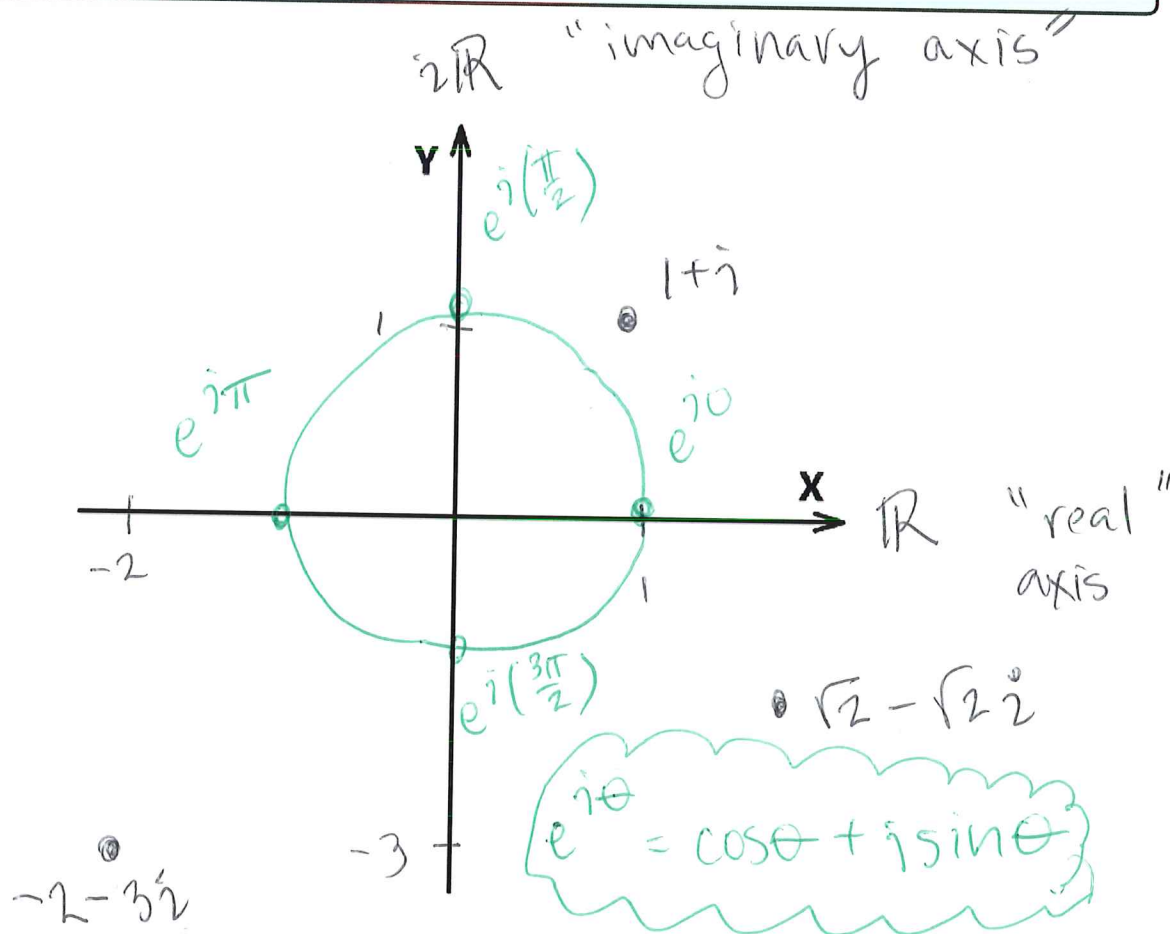

Activity: Plot Some Points

09:43

(2 min.)

Plot the following points on the complex plane.

1. $1 + i$
2. $-2 - 3i$
3. $\sqrt{2} - \sqrt{2}i$


Remark: What's Euler Saying?

Euler's identity is *really* saying that $e^{i\theta}$ traces out the unit circle in the complex plane.

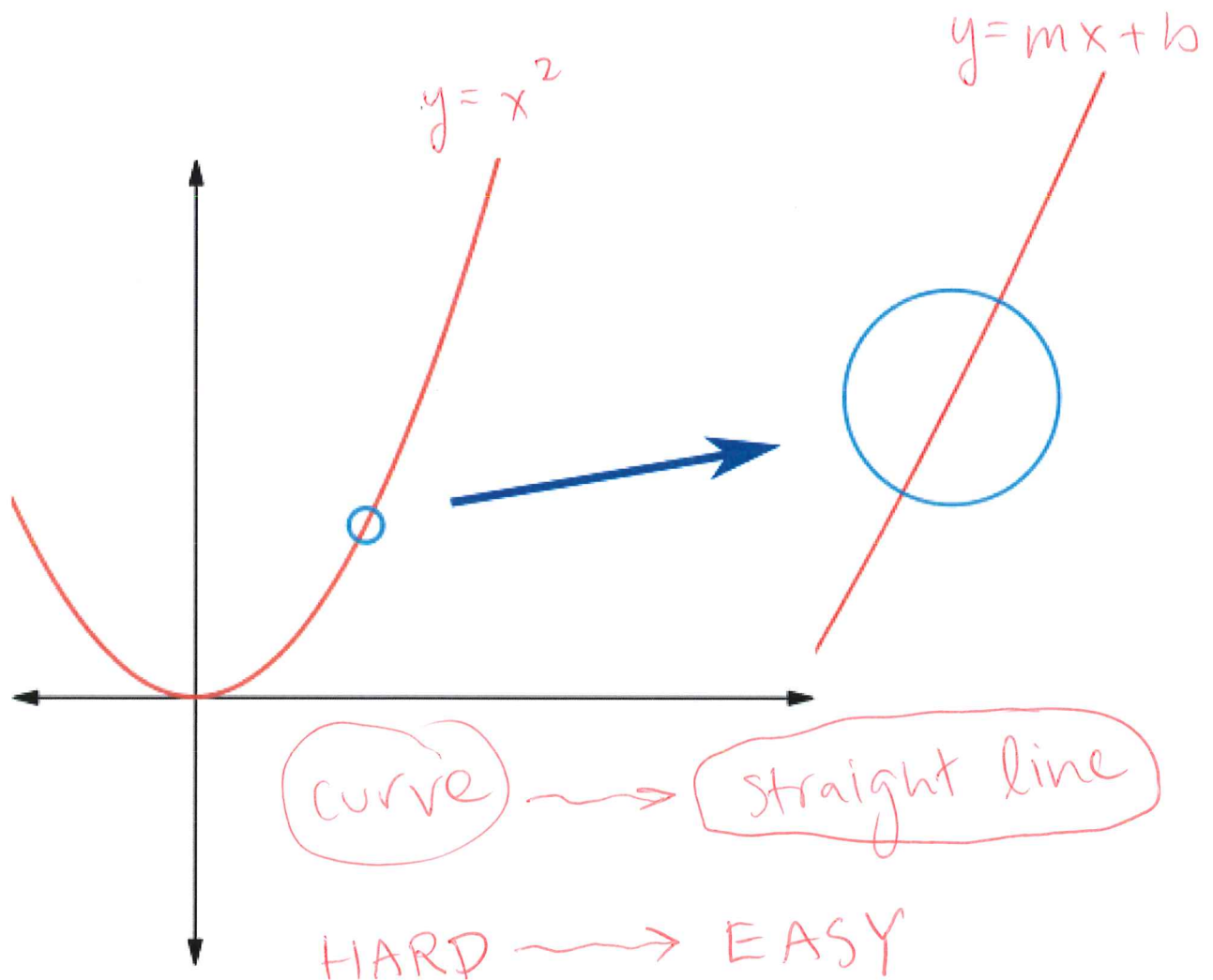
Section 2: Calculus

Remark: What is Calculus?

Calculus is a set of tools for understanding how things change. We are familiar with the notions of speed and velocity from physics. Calculus is the formal mathematical study of such concepts. For the sake of brevity, we focus on: polynomials, exponentials, and trigonometric functions.

Example: A Preview of Calculus

If we zoom in very close on any “nice” graph, we get a line.



Definition: Derivative / Slope

The derivative or slope of a function $f(x)$ at a point $x = a$ is:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Example: Compute the slope of a line.

What's the slope of $f(x) = mx + b$? $\rightarrow m$

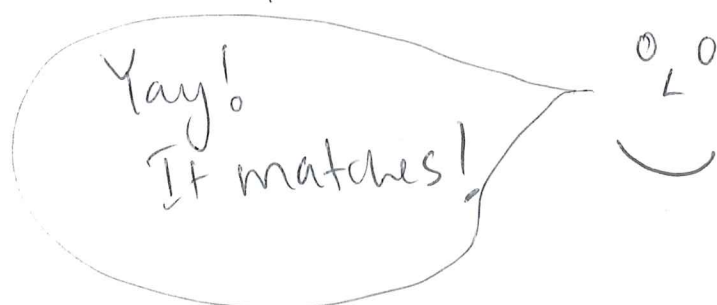
given $f(x) = mx + b$ we use the definition of slope:

$$f'(a) = \lim_{x \rightarrow a} \frac{(mx+b) - (ma+b)}{x-a}$$

$$= \lim_{x \rightarrow a} \frac{mx + \cancel{b} - ma - \cancel{b}}{x-a}$$

$$= \lim_{x \rightarrow a} \frac{m(\cancel{x-a})}{\cancel{x-a}} = \lim_{x \rightarrow a} m = m$$

Yay!
It matches!



**Activity: Slope of a Cubic**

10:00

(5 min.)

What's the slope of $f(x) = x^3$? How do we calculate it? $\rightarrow 3a^2$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{\cancel{(x-a)}(x^2 + ax + a^2)}{\cancel{x-a}}$$

$$= \lim_{x \rightarrow a} x^2 + ax + a^2$$

$$= a^2 + a^2 + a^2 = 3a^2 \checkmark$$

Theorem: Slope of Monomials

If n is a whole number, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Theorem: The Fundamental Trigonometric Limits

The slope of $y = \sin(x)$ at $x = 0$ is one. Formally, written as a limit,

$$\lim_{x \rightarrow 0} \frac{\sin(x) - \sin(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1.$$

Similarly, the slope of $y = \cos(x)$ at $x = 0$ is zero.

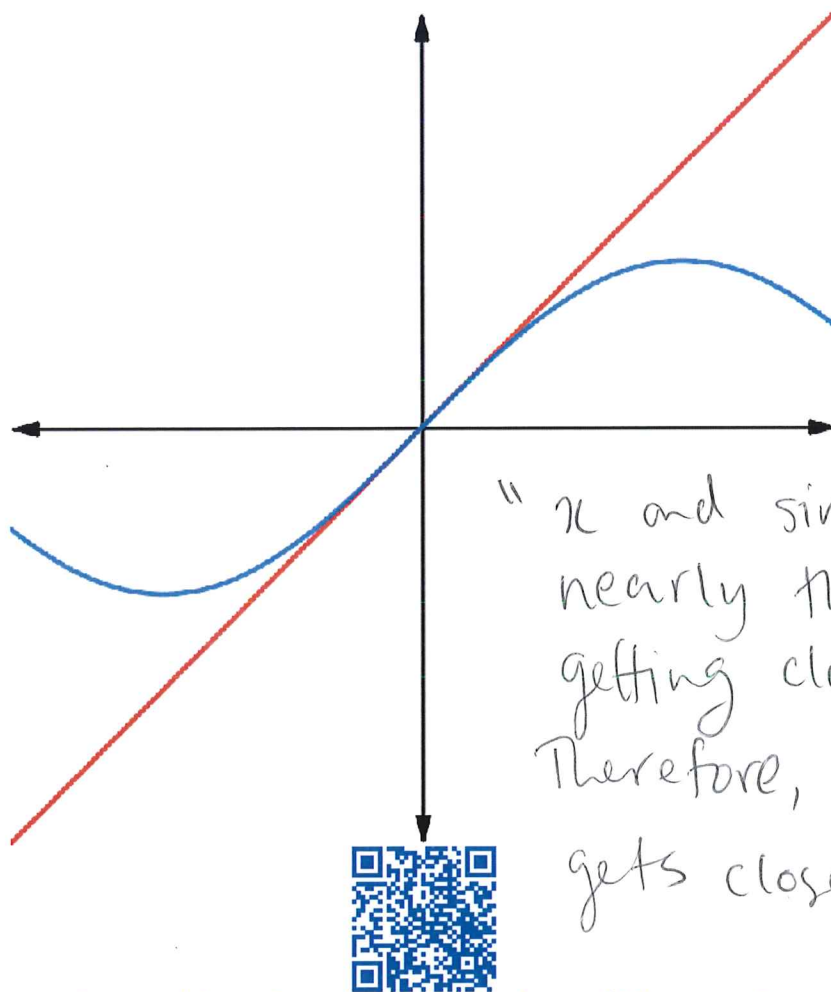
$$\lim_{x \rightarrow 0} \frac{\cos(x) - \cos(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = 0.$$

**Activity: Investigate A Limit Graphically**

10:08

(2 min.)

Use Desmos to graph $y = \sin(x)$ and $y = x$ on the same pair of axes and zoom in on the point $(0, 0)$. Write a paragraph describing what happens and how it relates to $f(x) = \frac{\sin(x)}{x}$.



<https://www.desmos.com/calculator/dfbwtcocof>

Theorem: The Fundamental Trigonometric Identities

We have the following:

$$1. \sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)$$

$$2. \cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$$

Note: Later, in the afternoon workshop, we will prove these as consequences of Euler's identity.

Example: The Derivative of $\sin(x)$.

Find the derivative of $f(x) = \sin(x)$.

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x)[\cos(h) - 1] + \cos(x)\sin(h)}{h} \\
 &= \sin(x) \lim_{h \rightarrow 0} \underbrace{\left[\frac{\cos(h) - 1}{h} \right]}_{=0} + \cos(x) \lim_{h \rightarrow 0} \underbrace{\left[\frac{\sin(h)}{h} \right]}_{=1} \\
 &= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x)
 \end{aligned}$$

**Activity: The Derivative of $\cos(x)$.**

10:21

(5 min.)

Find the derivative of $f(x) = \cos(x)$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos(x)[\cos(h) - 1] - \sin(x)\sin(h)}{h}$$

$$= \cos(x) \cdot 0 - \sin(x) \cdot 1 = -\sin(x)$$

Theorem: The Exponential Function

The exponential function is $f(x) = e^x$. It has two crucial properties:

1. $f(0) = 1. \rightarrow e^0 = 1$
2. $f'(x) = f(x) \rightarrow e^x$ is its own derivative

These properties imply $f(x) = e^x$.

Definition: Higher Derivatives

If $f(x)$ is a function then its derivative $f'(x)$ is also a function. This allows us to define **higher derivatives**. The “zero'th” derivative of $f(x)$ is itself: $f^{(0)}(x) = f(x)$. The n 'th derivative of $f(x)$, written $f^{(n)}(x)$, is the derivative of $f^{(n-1)}(x)$.

Example: The Derivatives of the Exponential

Find all the derivatives $f^{(k)}(x)$ of $f(x) = e^x$.

$$f^{(0)}(x) = f(x) = e^x$$

$$f^{(1)}(x) = f'(x) = e^x$$

$$f^{(2)}(x) = e^x$$

We get:

$$f^{(k)}(x) = e^x$$

for all $k = 0, 1, 2, \dots$

Example: Derivatives of a cubic.

Find all the derivatives $f^{(k)}(x)$ of $f(x) = x^3$.

$$f^{(0)}(x) = f(x) = x^3$$

$$f^{(1)}(x) = f'(x) = 3x^2$$

$$f^{(2)}(x) = 6x$$

$$f^{(3)}(x) = 6$$

We get:

$$f^{(k)}(x) = \begin{cases} x^3 & k=0 \\ 3x^2 & k=1 \\ 6x & k=2 \\ 6 & k=3 \\ 0 & k \geq 4 \end{cases}$$

$$f^{(4)}(x) = 0 \dots$$

Example: Derivatives of $\sin(x)$ and $\cos(x)$.

Find all the derivatives $f^{(k)}(x)$ and $g^{(k)}(x)$ for $f(x) = \sin(x)$ and $g(x) = \cos(x)$.

Notice that the pattern repeats. How many terms does it take for the pattern to repeat?

k	$f^{(k)}(x)$	$g^{(k)}(x)$
0	$\sin(x)$	$\cos(x)$
1	$\cos(x)$	$-\sin(x)$
2	$-\sin(x)$	$-\cos(x)$
3	$-\cos(x)$	$\sin(x)$
4	$\sin(x)$	$\cos(x)$
5	$\cos(x)$	$-\sin(x)$
6	$-\sin(x)$	$-\cos(x)$
7	$-\cos(x)$	$\sin(x)$

After four rows the pattern repeats!

This is "like" $z_n = i^n$.

Section 3: Designer Polynomials, Infinite Series, and Euler's Identity

Activity 10:40 (5 min) + 2

Example: Find The Derivatives at Zero

Consider the following polynomial,

$$p(x) = \boxed{3} + \boxed{5}x + \boxed{7}x^2 + \boxed{11}x^3 + \boxed{13}x^4 + \boxed{17}x^5$$

Compute the derivatives $p^{(k)}(0)$ of $p(x)$.

Note: This is just a normal polynomial with rational coefficients. The boxes are just for emphasis.

$$p^{(0)}(0) = p(0) = 3 + \frac{5}{1} \cdot 0 + \frac{7}{2} \cdot 0^2 + \frac{11}{6} \cdot 0^3 + \frac{13}{24} \cdot 0^4 + \frac{17}{120} \cdot 0^5 = \boxed{3}$$

$$p^{(1)}(x) = p'(x) = 0 + \frac{5}{1} \cdot 1 + \frac{7}{2} \cdot 2x + \frac{11}{6} \cdot 3x^2 + \frac{13}{24} \cdot 4x^3 + \frac{17}{120} \cdot 5x^4$$

$$p^{(1)}(0) = \frac{5}{1} \cdot 1 = \boxed{5}$$

$$p^{(2)}(x) = 0 + \frac{7}{2} \cdot 2 + \frac{11}{6} \cdot 3 \cdot 2x + \frac{13}{24} \cdot 4 \cdot 3 \cdot x^2 + \frac{17}{120} \cdot 5 \cdot 4 \cdot x^3$$

$$p^{(2)}(0) = \frac{7}{2} \cdot 2 = \boxed{7}$$

$$p^{(3)}(x) = \frac{11}{6} \cdot 3 \cdot 2 \cdot 1 + \frac{13}{24} \cdot 4 \cdot 3 \cdot 2x + \frac{17}{120} \cdot 5 \cdot 4 \cdot 3 \cdot x^2$$

$$p^{(3)}(0) = \frac{11}{6} \cdot 3 \cdot 2 \cdot 1 = \boxed{11}$$

Exercise:

$p^{(4)}(0)$ and $p^{(5)}(0)$

Remark: This idea generalizes!

Notice that we could put any numbers whatsoever in the boxes. Given any finite sequence of numbers a_k we can make a polynomial $p(x)$ such that: $p^{(k)}(0) = a_k$. Let's make a theorem out of this!

Definition: Factorials

The **factorial** of n is $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. For example: $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

Theorem: "Designer" Polynomials

Given a finite sequence a_k for $k = 0, 1, \dots, n$. Consider the polynomial

$$p(x) = a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \cdots + \frac{a_n}{n!}x^n.$$

This polynomial satisfies $p^{(k)}(x) = a_k$ for $k = 0, 1, \dots, n$.

Alternatively, we define $n!$ by

$$f(x) = x^n \Rightarrow f^{(n)}(0) = n!$$

For $k=0$ we have:

$$p(0) = p^{(0)}(0) = a_0 + \frac{a_1}{1!} \cancel{0} + \cdots + \frac{a_n}{n!} 0^n = \boxed{a_0}$$

For $k=1$ we have:

$$p^{(1)}(0) = 0 + \frac{a_1}{1!} 1 + \underbrace{\left(\frac{a_2}{2!} 2 \cancel{0} + \cdots + \frac{a_n}{n!} n \cancel{0}^{n-1} \right)}_{=0} = 0 + a_1 = \boxed{a_1}$$

For $k=2$ we have:

$$p^{(2)}(0) = 0 + \frac{a_2}{2!} 2 \cdot 1 + \underbrace{\left(\frac{a_3}{3!} 3 \cdot 2 \cdot \cancel{0} + \cdots \right)}_{=0}$$

$$= \boxed{a_2}$$

Remark: The Exponential's Derivatives

Recall, we previously calculated: if $f(x) = e^x$ then $f^{(k)}(0) = 1$.
That is: All the derivatives of the exponential are one at $x = 0$.

Theorem: Approximating the Exponential Function

If n is very large, then

$$e^x \approx 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n.$$

Hard ↗

↖ Easy

We chose the coefficients $a_k = f^{(k)}(0) = 1$.
This is a polynomial with the same derivatives as $f(x) = e^x$.

What does the calculator DO when you hit the e^x button? Evaluates a polynomial!
 $n=10$

$$n=0 \rightarrow f(x) = 1$$

$$n=1 \rightarrow f(x) = 1 + \frac{1}{1!}x = 1 + x$$

$$n=2 \rightarrow f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 = 1 + x + \frac{x^2}{2}$$



<https://www.desmos.com/calculator/wtalombu2p>

Example: Approximate $\sin(x)$ If n is very large, then

$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \dots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

↗ This is a polynomial with the same derivatives as $\sin(x)$.

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\sin(x)$	$\sin(0) = 0$
1	$\cos(x)$	$\cos(0) = 1$
2	$-\sin(x)$	$-\sin(0) = 0$
3	$-\cos(x)$	$-\cos(0) = -1$

odd values of k .

In general, the $\sin(x)$ function satisfies

~~$$f^{(2n)}(0) = (-1)^n$$~~

$$f^{(2n+1)}(0) = (-1)^n$$



<https://www.desmos.com/calculator/690dbrl3zs>

**Activity: Approximate $\cos(x)$**

11:14

(5 min)

If n is very large, then

$$\cos(x) \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots + \frac{(-1)^n}{(2n)!}x^{2n}.$$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\cos(x)$	$\cos(0) = 1$
1	$-\sin(x)$	$-\sin(0) = 0$
2	$-\cos(x)$	$-\cos(0) = -1$
3	$\sin(x)$	$\sin(0) = 0$

Only even terms.

we get $f^{(2n)}(0) = (-1)^n.$


<https://www.desmos.com/calculator/rn4lpyocex>

Remark: Summary of Approximations

$$e^x \approx 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}x^9$$

$$\cos(x) \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8$$

$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9$$

Theorem: Euler's Identity

Plugging $x = i\theta$ in to the approximations above gives the famous $e^{i\theta} = \cos(\theta) + i \sin(\theta)$.

$$\begin{aligned} e^{i\theta} &\approx 1 + \frac{1}{1!}(i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 \\ &\quad + \frac{1}{5!}(i\theta)^5 + \frac{1}{6!}(i\theta)^6 + \frac{1}{7!}(i\theta)^7 + \frac{1}{8!}(i\theta)^8 \\ &\approx \left[1 + \frac{1}{1!}(i\theta) - \frac{1}{2!}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 \right. \\ &\quad \left. + \frac{1}{5!}i\theta^5 - \frac{1}{6!}\theta^6 - \frac{1}{7!}i\theta^7 + \frac{1}{8!}\theta^8 \right] \\ &\approx \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \frac{1}{8!}\theta^8 \right) \\ &\quad + \left(\frac{1}{1!}\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 \right) i \\ &\approx \cos(\theta) + i \sin(\theta) \end{aligned}$$