Section 1: Complex Numbers

Remark: Where are we going? How will we get there?

The goal of this course is to get to Euler's theorem. We are going to do a very brief introduction to complex numbers and calculus. The class will be taught in an active learning style with lots of tasks. Please feel free to ask any questions. I hope that you enjoy the adventure.

Theorem: Euler's Theorem

If θ is a real number, then

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Activity: Which value? Why? (2 min.)

Find a value to plug in for θ to obtain:

$$e^{i\pi} + 1 = 0.$$

Re-arrange Euler's equation to obtain this one.

Activity: Name the parts. (2 min.)

What are the constants e, i, π, 0, and 1? How are they defined?

Definition: The Real and Complex Numbers

The **real numbers** are the familiar numbers expressible as decimals. We write $x \in \mathbb{R}$ for the statement "x is in the real numbers". The **complex numbers** are numbers of the form $a + bi$ where a and b are both real numbers. The **imaginary unit** i is defined so that: $i^2 = -1$.

Activity: Calculate some values. (5 min.)

Calculate the value of the following expressions.

1. $(1-i)(1+i)$
2. $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

Example: Powers of the Imaginary Unit

Consider the sequence of complex numbers $z_n = i^n$.
Write a simple pattern for the entries of this sequence.
Notice that the pattern repeats. How many terms does it take for the pattern to repeat?

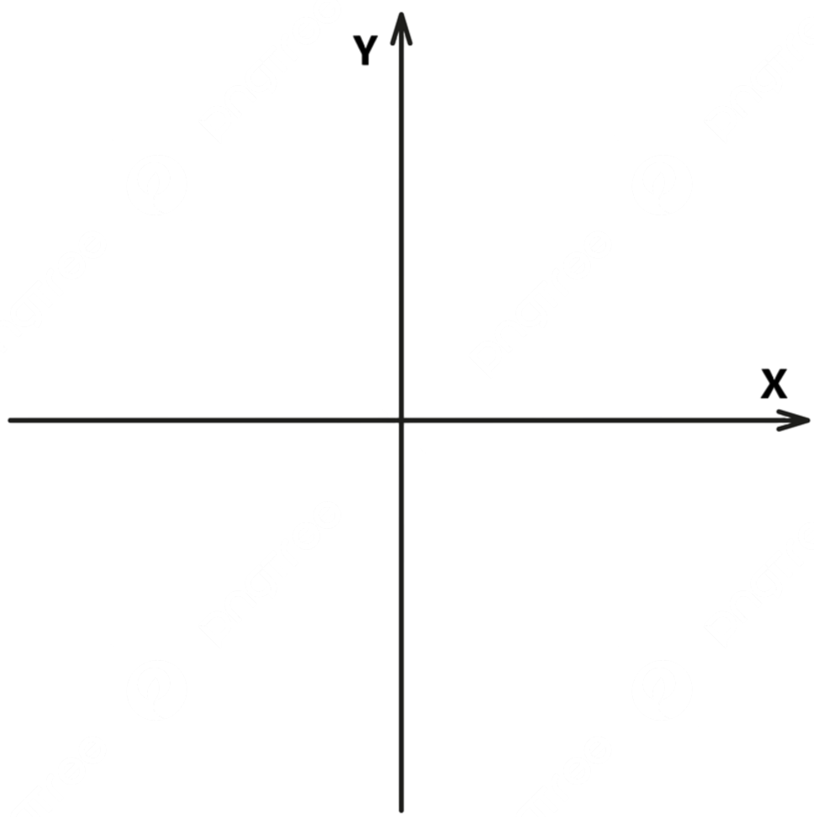
Definition: The Complex Plane

The **complex plane** $\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$ has a **real axis** $\mathbb{R}$ and an **imaginary axis** $i\mathbb{R}$.

Activity: Plot Some Points **(2 min.)**

Plot the following points on the complex plane.

1. $1 + i$
2. $-2 - 3i$
3. $\sqrt{2} - \sqrt{2}i$



Remark: What's Euler Saying?

Euler's identity is *really* saying that $e^{i\theta}$ traces out the unit circle in the complex plane.

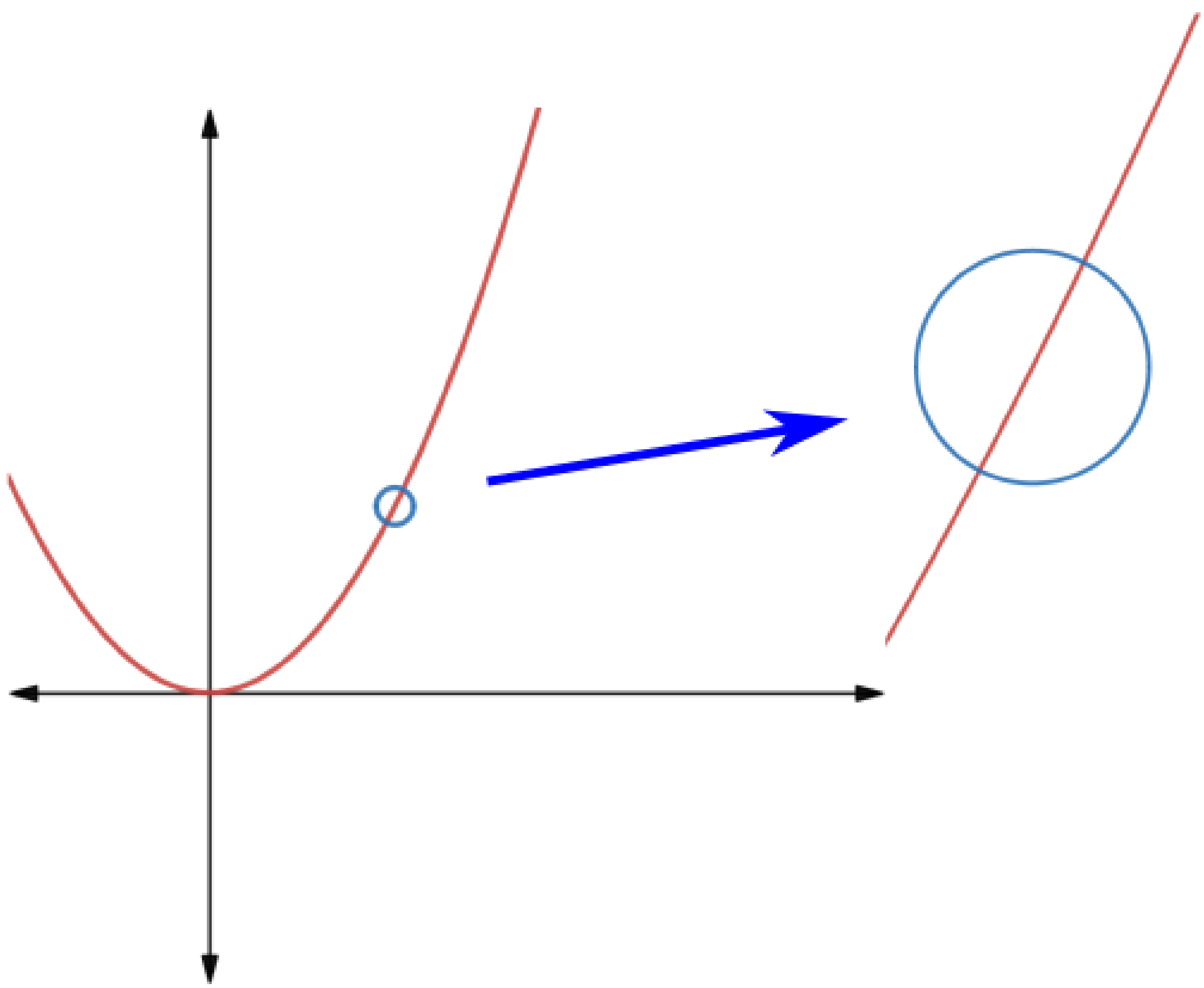
Section 2: Calculus

Remark: What is Calculus?

Calculus is a set of tools for understanding how things change. We are familiar with the notions of speed and velocity from physics. Calculus is the formal mathematical study of such concepts. For the sake of brevity, we focus on: polynomials, exponentials, and trigonometric functions.

Example: A Preview of Calculus

If we zoom in very close on any "nice" graph, we get a line.



Definition: Derivative / Slope

The **derivative** or **slope** of a function $f(x)$ at a point $x = a$ is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: Compute the slope of a line.

What's the slope of $f(x) = mx + b$?

Activity: Slope of a Cubic (5 min.)

What's the slope of $f(x) = x^3$? How do we calculate it?

Theorem: Slope of Monomials

If n is a whole number, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Theorem: The Fundamental Trigonometric Limits

The slope of $y = \sin(x)$ at $x = 0$ is one. Formally, written as a limit,

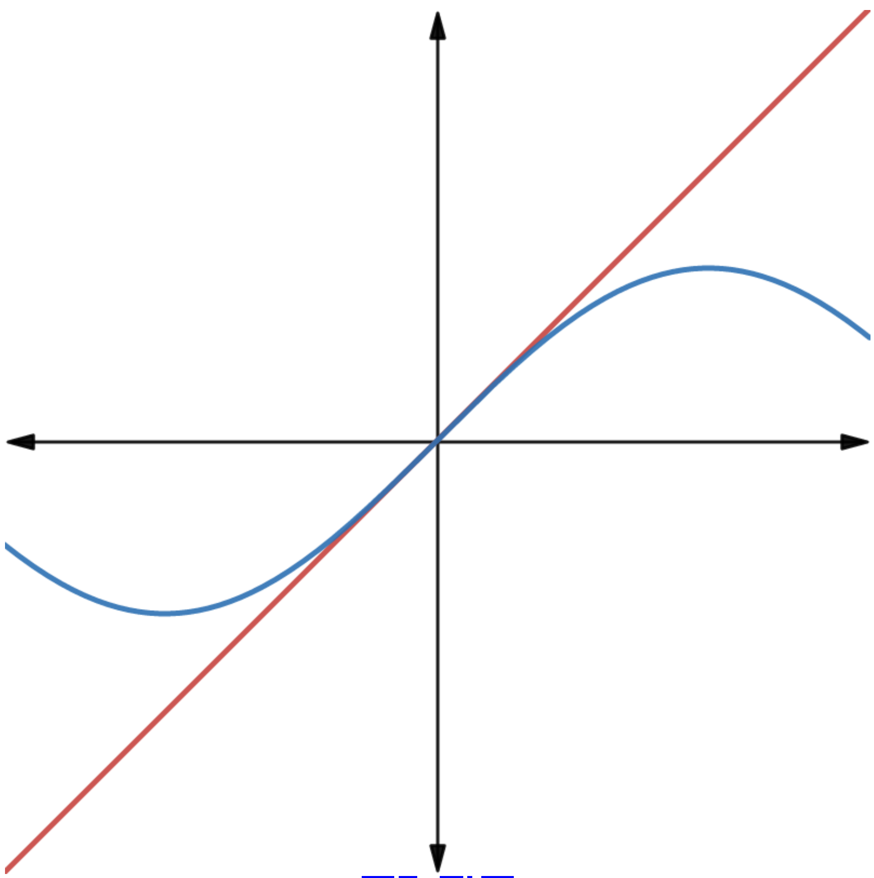
$$\lim_{x\to 0} \frac{\sin(x) - \sin(0)}{x - 0} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Similarly, the slope of $y = \cos(x)$ at $x = 0$ is zero.

$$\lim_{x\to 0} \frac{\cos(x) - \cos(0)}{x - 0} = \lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0.$$

Activity: Investigate A Limit Graphically (2 min.)

Use Desmos to graph $y = \sin(x)$ and $y = x$ on the same pair of axes and zoom in on the point $(0, 0)$. Write a paragraph describing what happens and how it relates to $f(x) = \frac{\sin(x)}{x}$.



https://www.desmos.com/calculator/dfbwtcocof

Theorem: The Fundamental Trigonometric Identities

We have the following:

1. $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
2. $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

Note: Later, in the afternoon workshop, we will prove these as consequences of Euler's identity.

Example: The Derivative of $\sin(x)$.

Find the derivative of $f(x) = \sin(x)$.

Activity: The Derivative of $\cos(x)$. (5 min.)

Find the derivative of $f(x) = \cos(x)$.

Theorem: The Exponential Function

The exponential function is $f(x) = e^x$. It has two crucial properties:

1. $f(0) = 1$.
2. $f'(x) = f(x)$

These properties imply $f(x) = e^x$.

Definition: Higher Derivatives

If $f(x)$ is a function then its derivative $f'(x)$ is also a function. This allows us to define **higher derivatives**. The **"zero'th" derivative** of $f(x)$ is itself: $f^{(0)}(x) = f(x)$. The n**'th derivative** of $f(x)$, written $f^{(n)}(x)$, is the derivative of $f^{(n-1)}(x)$.

Example: The Derivatives of the Exponential

Find all the derivatives $f^{(k)}(x)$ of $f(x) = e^x$.

Example: Derivatives of a cubic.

Find all the derivatives $f^{(k)}(x)$ of $f(x) = x^3$.

Example: Derivatives of $\sin(x)$ and $\cos(x)$.

Find all the derivatives $f^{(k)}(x)$ and $g^{(k)}(x)$ for $f(x) = \sin(x)$ and $g(x) = \cos(x)$.
Notice that the pattern repeats. How many terms does it take for the pattern to repeat?

Section 3: Designer Polynomials, Infinite Series, and Euler's Identity

Activity: Find The Derivatives at Zero (5 min.)

Consider the following polynomial,

$$p(x) = \boxed{3} + \frac{\boxed{5}}{1}x + \frac{\boxed{7}}{2}x^2 + \frac{\boxed{11}}{6}x^3 + \frac{\boxed{13}}{24}x^4 + \frac{\boxed{17}}{120}x^5$$

Compute the derivatives $p^{(k)}(0)$ of $p(x)$.
Note: This is just a normal polynomial with rational coefficients. The boxes are just for emphasis.

Remark: This idea generalizes!

Notice that we could put any numbers whatsoever in the boxes. Given any finite sequence of numbers a_k we can make a polynomial $p(x)$ such that: $p^{(k)}(0) = a_k$. Let's make a theorem out of this!

Definition: Factorials

The **factorial of** n is $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. For example: $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

Theorem: "Designer" Polynomials

Given a finite sequence a_k for $k = 0, 1, \ldots, n$. Consider the polynomial

$$p(x) = a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \cdots + \frac{a_n}{n!}x^n.$$

This polynomial satisfies $p^{(k)}(x) = a_k$ for $k = 0, 1, \ldots, n$.

Remark: The Exponential's Derivatives

Recall, we previously calculated: if $f(x) = e^x$ then $f^{(k)}(0) = 1$.
That is: All the derivatives of the exponential are one at $x = 0$.

Theorem: Approximating the Exponential Function

If n is very large, then

$$e^x \approx 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n.$$

https://www.desmos.com/calculator/wtalombu2p

Example: Approximate $\sin(x)$

If n is very large, then

$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

https://www.desmos.com/calculator/690dbrl3zs

Activity: Approximate $\cos(x)$ **(5 min.)**

If n is very large, then

$$\cos(x) \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n}.$$

https://www.desmos.com/calculator/rn4lpyocex

Remark: Summary of Approximations

$$
\begin{array}{rcccccccccccc}
e^x & \approx & 1 & +\frac{1}{1!}x & +\frac{1}{2!}x^2 & +\frac{1}{3!}x^3 & +\frac{1}{4!}x^4 & +\frac{1}{5!}x^5 & +\frac{1}{6!}x^6 & +\frac{1}{7!}x^7 & +\frac{1}{8!}x^8 & +\frac{1}{9!}x^9 \\
\cos(x) & \approx & 1 & & -\frac{1}{2!}x^2 & & +\frac{1}{4!}x^4 & & -\frac{1}{6!}x^6 & & +\frac{1}{8!}x^8 & \\
\sin(x) & \approx & & x & & -\frac{1}{3!}x^3 & & +\frac{1}{5!}x^5 & & -\frac{1}{7!}x^7 & & +\frac{1}{9!}x^9
\end{array}
$$

Theorem: Euler's Identity

Plugging $x = i\theta$ in to the approximations above gives the famous $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.