Plotting Practice

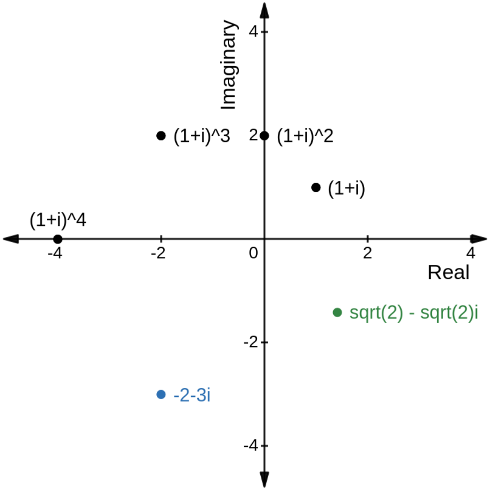
Question

Plot the following complex numbers.

1. $1 + i$
2. $-2 - 3i$
3. $\sqrt{2} - \sqrt{2}i$

Plot the series $z_n = (1 + i)^n$.

Plotting Practice (Solutions)



Calculation Practice

Question

Calculate the following values.

1. $(1+i)^4$
2. $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2$

$$
\begin{aligned}
(1+i)^4 &= 1^4 + 4 \cdot 1^3 \cdot i + 6 \cdot 1^2 \cdot i^2 + 4 \cdot 1 \cdot i^3 + i^4 \\
&= 1 + 4i - 6 - 4i + 1 = -4 \\
\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 &= \left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}i\right) + \left(\frac{1}{\sqrt{2}}i\right)^2 \\
&= \frac{1}{2} + i - \frac{1}{2} = -i
\end{aligned}
$$

Length

Question

*Calculating a Length The **complex conjugate** of $a + bi$ is $\overline{a + bi} = a - bi$. We define the **magnitude**of z by $|z|^2 = z\bar{z}$. Alternatively, we call $|z|$ the **length** of z.*
Compute the length $|3 + 4i|$.

We have the following:

$$|3 + 4i|^2 = (3 + 4i)(3 - 4i) = 3^2 + 4^2 \implies |3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Conjugates and Products

Question

If z and w are complex numbers then $\overline{zw} = \overline{z}\ \overline{w}$. This implies $|zw| = |z||w|$.

Suppose $z = a + bi$ and $w = c + di$. We compute both sides of the equation: $\overline{zw} = \overline{z}\ \overline{w}$. The left hand side is:

$$\overline{(a+bi)(c+di)} = \overline{ac + adi + bci - bd} = \overline{(ac-bd) + (ad+bc)i} = (ac-bd) - (ad+bc)i$$

The other side of the equation is:

$$\overline{(a+bi)}\ \overline{(c+di)} = (a-bi)(c-di) = ac - adi - bci - bd = (ac-bd) - (ad+bc)i$$

As these are equal, we've proved: $\overline{zw} = \overline{z}\ \overline{w}$.

Complex Roots

Question

Factor the polynomials $f(x) = x^2 + 1$ *and* $g(x) = x^3 - 1$ *over the complex numbers.*

We use the fact that $i^2 = -1$ to calculate:

$$f(x) = x^2 + 1 = x^2 - (i)^2 = (x - i)(x + i).$$

For $g(x) = x^3 - 1$, we have a real roots and so:

$$g(x) = (x-1)(x^2 + x + 1) = (x - 1)\left(x - \left(\frac{-1 - \sqrt{3}i}{2}\right)\right)\left(x - \left(\frac{-1 + \sqrt{3}i}{2}\right)\right).$$

We used the quadratic equation to solve for $x^2 + x + 1 = 0$.

Prove Trigonometric Identities

Question

Suppose that Euler's identity is true. Prove the following trigonometric identities.

1. $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
2. $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

We handle both equations simultaneously.

$$\begin{aligned}
e^{(a+b)i} &= e^{ai}e^{bi} \\
&= (\cos(a) + i\sin(a))\,(\cos(b) + i\sin(b)) \\
&= \cos(a)\sin(b) + \cos(a)\sin(b)i + \sin(a)\cos(b)i - \sin(a)\sin(b) \\
&= [\cos(a)\cos(b) - \sin(a)\sin(b)] + [\cos(a)\sin(b) + \sin(a)\cos(b)]\,i
\end{aligned}$$

Euler's identity also guarantees: $e^{(a+b)i} = \cos(a+b) + i\sin(a+b)$.
We equate both expressions for $e^{(a+b)i}$ and get the desired trigonometric equations.

Geometric Series

Question

1. *If $z \neq 1$ is a complex number and n is a whole number,*

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^n}{1 - z}.$$

2. *Suppose $|z| < 1$. As n tends to infinity, we get:*

$$1 + z + z^2 + \cdots = \frac{1}{1 - z}.$$

3. *Use this fact to tell a story about pouring two cups of tea.*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

Geometric Series (Solutions)

Question

If $z \neq 1$ is a complex number and n is a whole number,

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^n}{1 - z}.$$

To show the desired equality, we multiply $1 + z + z^2 + \cdots + z^n$ by $(1 - z)$.

$$\begin{aligned}(1 + z + z^2 + \cdots + z^n)(1 - z) &= (1 + z + z^2 + \cdots + z^n) \\ &\quad - z(1 + z + z^2 + \cdots + z^n) \\ &= (1 + z + z^2 + \cdots + z^n) \\ &\quad - z - z^2 - z^3 + \cdots - z^{n+1} = 1 - z^{n+1}\end{aligned}$$

We then have:

$$(1 + z + z^2 + \cdots + z^n)(1 - z) = 1 - z^{n+1} \Longrightarrow 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

Geometric Series (Solutions)

Question

Suppose $|z| < 1$. *As n tends to infinity, we get:*

$$1 + z + z^2 + \cdots = \frac{1}{1-z}.$$

From the previous part, we have:

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1-z}$$

If $|z| < 1$ then $|z|^{n+1}$ decreases to zero as a functon of n. And so, we can make $|z|^{n+1}$ as small as we want. If n gets very large, $z^{n+1} \to 0$. This gives:

$$1 + z + z^2 + \cdots = \frac{1-0}{1-z} = \frac{1}{1-z}$$

Geometric Series (Solutions)

Question

Use this fact to tell a story about pouring two cups of tea.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

Two friends walk into a tea shop and ask for tea. The owner pours a cup of tea, and leaves the second cup empty. $(+1)$ *The second friend says: "Hey! You didn't pour any tea in my cup." The owner pours a half cup of tea.* $(+1/2)$ *The second friend says: "That's not a whole cup! Please give me more tea." The owner pours another quarter cup of tea.* $(+1/4)$

This goes on for a long time...

An Alternative Proof of Euler

Question

We need a couple other facts from calculus.
If a function $f(x)$ is constant, then $f'(x) = 0$.

$$f(x) = e^{ix} \implies f'(x) = ie^{ix} \qquad h'(x) = \frac{f(x)}{g(x)} \implies h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

We can use these facts to prove Euler's identity.

To prove Euler's identity, we show that the following is constant:

$$h(x) = \frac{\cos(x) + i\sin(x)}{e^{ix}}$$

We calculate:

$$h(x) = \frac{(\cos(x) + i\sin(x))'e^{ix} - (\cos(x) + i\sin(x))(e^{ix})'}{(e^{ix})^2}$$

An Alternative Proof of Euler (Solutions)

We calculate:

$$\begin{aligned} h(x) &= \frac{(\cos(x) + i\sin(x))' e^{ix} - (\cos(x) + i\sin(x))(e^{ix})'}{(e^{ix})^2} \\ &= \frac{(-\sin(x) + i\cos(x))e^{ix} - (\cos(x) + i\sin(x))(ie^{ix})}{(e^{ix})^2} \\ &= \frac{[-\sin(x) + i\cos(x) - i\cos(x) + \sin(x)]\, e^{ix}}{(e^{ix})^2} = \frac{0}{(e^{ix})^2} = 0 \end{aligned}$$

Therefore $h'(x) = 0$ for all x. It follows that $h(x)$ is constant.
If we plug in $x = 0$, we get:

$$h(0) = \frac{\cos(0) + i\sin(0)}{e^{0i}} = \frac{1}{1} = 1$$